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Data structures and algorithms for topological analysis

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***Abstract*—One of the steps of geometric modeling is to know the topology and/or the geometry of the objects considered. This paper presents different data structures and algorithms used in this study. We are particularly interested by algebraic structures, eg homotopy and homology groups, the Betti numbers, the Euler characteristic, or the Morse-Smale complex. We have to be able to compute these data structures, and for (homotopy and homology) groups, we also want to compute their generators. We are also interested in algorithms CIA and HIA presented in the thesis of Nicolas DELANOUE, which respectively compute the connected components and the homotopy type of a set defined by a CSG (constructive solid geometry) tree. We would like to generalize these algorithms to sets defined by projection.**

I. INTRODUCTION

The main problem of computional topology is due to the floating calculus. Indeed, we cannot decide if a space is open or closed which is the basis notions in topology (we say that topology is the study of neighborhoods). Then with floating computation, we are not be able to decide if two spaces are homeomorphic. That's why, we consider topological data structures, mostly algebraic, because their computation is based on integer calculus and therefore on an exact computation.

Data structures and algorithms that we give here are topological invariants (ie invariant under homeomorphism) so they are used to classify topological spaces.

They have also a topological interpretation (sometimes also geometrically) which may be useful for the geometric modeling: for example, some of these data compute the (path-)connected components, the number of holes of a certain dimension, ...

We want to compute these data structures for geometric sets defined in different ways:

- implicitly: CSG trees where bases are sets of primitive

- parametrically: varying parameters in the square

but also projections, Minkowski sums,... and boolean operations of such sets. We also have to treat meshs, smooth and piecewise linear manifolds, simplicial complex,...

II. HOMOLOGY AND HOMOTOPY GROUPS [2] [1]

Background: A *topological invariant* is a map f that assigns the same object to homeomorphic topological spaces, that is to say: $X \simeq Y \Rightarrow f(X) = f(Y)$.

Examples: The number of (path-)connected components, all topological properties (compactedness, connectedness,...).

The idea of homotopy groups and homology and algebraic topology in general is to bring a topological problem (Are two spaces homeomorphic ?) to an algebraic problem (Are two groups isomorphic ?), and this last problem is more easier than the topological one. We know that these groups are topological invariants.

A. Homotopy groups

In this section, we first give the example of the first homotopy group: the fundamental group (sometimes called the Poincaré group), and the homotopy groups of upper order.

The *fundamental group* of a pointed topological space (X, x) is the set of homotopy classes of loops in X which rely x to itself ($x \in X$), provided by the composition of paths. We note it by: $\Pi_1(X, x)$. If X is path-connected, then the fundamental group does not depend on the basis point and we will note $\Pi_1(X)$.

Examples:

- $\Pi_1(\mathbb{R}^m) = \{e\}$ (the trivial group).
- $\forall d \geq 2, \Pi_1(S^d) = \{e\}$ (S^d is the unit sphere of $\mathbb{R}^{d+1}$).
- $\Pi_1(S^1) = \mathbb{Z}$ ($\mathbb{Z}$ counts the number of loops around the circle).

Proposition 1 (Compatibility with product): Let (X, x) and (Y, y) two pointed topological spaces, then we have:

$$\Pi_1(X \times Y, (x, y)) = \Pi_1(X, x) \times \Pi_1(Y, y).$$

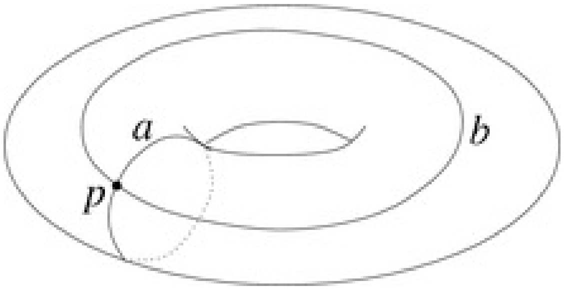

Fig. 1: The generators of $\Pi_1(T^2)$

Examples: The torus $T^2 := S^1 \times S^1$ in $\mathbb{R}^3$ is path-connected and: $\Pi_1(T^2) = \mathbb{Z}^2$.

The fundamental group is sometimes very useful to prove that two spaces are not homeomorphic. For example, the unit sphere S^2 and the torus T^2 in $\mathbb{R}^3$ are not homeomorphic since:

$$\{e\} \simeq \Pi_1(S^2) \neq \Pi_1(T^2) \simeq \mathbb{Z}^2$$

although these two spaces are compact, connected, and are closed 2-manifolds in $\mathbb{R}^3$.

The introduction of the fundamental group is quite natural, for against the actual computation of this data structure is very complex for any topological space.

Homotopy groups of upper order: We note $I^n := [0,1]^n$ and ∂I^n its boundary. These groups generalize the previous one. For $n \in \mathbb{N}$, the n^{th} homotopy group of (X, x_0) is the set of homotopy classes of maps $f : I^n \longrightarrow X$ such that $f(\partial I^n) = x_0$, provided by the composition. We note this group $\Pi_n(X, x_0)$. For $n \geq 2$, the n^{th} homotopy groups are commutative but they are not finitely generated and then we cannot determine easily their structure.

B. Homology groups

Unlike the homotopy groups, homology groups have a less obvious construction but their actual computation is much easier. We define in this section the notion of *simplicial homology*.

The simplicial homology is before defined for a *simplicial complex*.

Let $K = (\delta_i)_{i \in I}$ be a simplicial complex in $\mathbb{R}^n$.

Definition 1: The *Euler characteristic* of K is defined by:

$$\chi(K) := \sum_{j \in \mathbb{N}} (-1)^j n_j$$

where: $n_j := |\{i \in I \mid dim\ \delta_i = j\}|$.

Proposition 2: The Euler characteristic is a topological invariant.

We note (for all $n \in \mathbb{N}$) $C_n(K)$ the free group generated by the n-simplices of K.

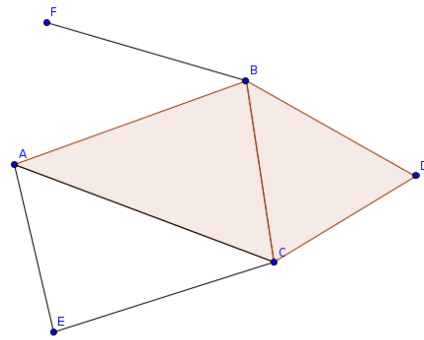

Fig. 2: A simplicial complex

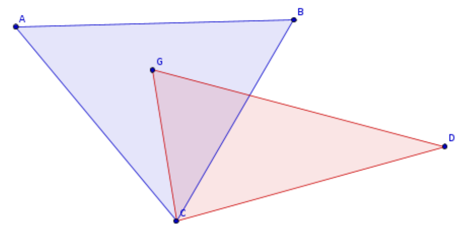

Fig. 3: counter-example of a simplicial complex

Definition 2: We call *boundary operators* and we note $\partial = (\partial_n : C_n(K) \longrightarrow C_{n-1}(K))_{n \in \mathbb{N}}$ the collection of group morphisms defined by:

$$\partial_n([v_0, ..., v_n]) = \sum_{i=0}^{n} (-1)^i [v_0, ..., \widehat{v_i}, ..., v_n].$$

So, we have a sequence of group morphisms:

$$\ldots \xrightarrow{\partial_{n+2}} C_{n+1}(K) \xrightarrow{\partial_{n+1}} C_n(K) \xrightarrow{\partial_n} \ldots$$

$$C_1(K) \xrightarrow{\partial_1} C_0(K) \xrightarrow{\partial_0 = 0} C_{-1}(K) = \{0\}.$$

with the property: $\forall\ n \in \mathbb{N},\ Im\ \partial_{n+1} \subset ker\ \partial_n$.

We call n-cycles the elements of $ker\ \partial_n$ and n-boundaries the elements of $Im\ \partial_{n+1}$.

Definition 3: Let K be a simplicial complex, we then define the *n-th homology group* of K by:

$$H_n(K) = ker\ \partial_n\ /\ Im\ \partial_{n+1}.$$

In order to extend the definition of homology groups on any topological spaces, we have to introduce the notion of Δ-*complex structure*.

The standard simplex of $\mathbb{R}^n$ is: $\Delta^n := \{(x_1, ..., x_n) \mid x_i \geq 0 \text{ et } \sum_{i=1}^{n} x_i = 1\}$.

Definition 4: A Δ-*complex structure* on a topological space X is a collection of continuous maps $(\sigma_\alpha)_{\alpha \in \Lambda}$ where $\sigma_\alpha : \Delta^{n_\alpha} \longrightarrow X$ such that:

(i) $\forall\ \alpha \in \Lambda$, the restriction $\sigma_\alpha | Int(\Delta^{n_\alpha})$ is injective
(ii) $\forall\ x \in X,\ \exists\ !\ \alpha \in \Lambda,\ x \in \sigma_\alpha(Int(\Delta^{n_\alpha}))$
(iii) $\forall\ U \subset X,\ (U$ is open in $X) \Longleftrightarrow$
$(\forall\ \alpha \in \Lambda,\ \sigma_\alpha^{-1}(U)$ is open in $\Delta^{n_\alpha})$.

Vocabulary: A topological space provided by a Δ-complex structure is also called a Δ-complex.

Remark: With these axioms, we have the following property: $\forall\ n \in \mathbb{N},\ \sigma_\alpha | Int(\Delta^{n_\alpha})$ is a homeomorphism on its image. Thus, we can say that the $\sigma_\alpha(Int(\Delta^{n_\alpha}))$ are open n_α-simplices on X, and that $\{\sigma_\alpha(Int(\Delta^{n_\alpha}))\}_{\alpha\in\Lambda}$ form a simplicial complex built on X (we also say *triangulation* of X).

Then we can define the homology groups of a Δ-complex considering for all n in $\mathbb{N}$, $\Delta_n(X)$ the free group generated by the open n-simplices of X. These homology groups will noted: $H_n^\Delta(X)$.

Remark: This definition of homology groups does not depend on the structure of Δ-complex defined on the topological space (we use the notion of *singular homology* to prove it). Thus, this data structure only depends on the topological space considered.

The $H_n^\Delta(X)$ are finitely generated, the structure theorem of finitely generated abelian groups gives us:

$$\exists\ !\ \beta_n \in \mathbb{N}^\star, \exists\ !\ m \in \mathbb{N}^\star, \exists\ !\ t_1 \mid t_2 \mid ... \mid t_m \geq 2,$$

$$H_n^\Delta(X) \simeq \prod_{i=1}^{\beta_n} \mathbb{Z} \times \prod_{j=1}^{m} \mathbb{Z}/t_j\mathbb{Z}$$

Definition 5: β_n is called the *n-th Betti number* of X.

Proposition 3: 1. β_0 is the number of connected components of X.
2. [Euler-Poincaré formula]:

$$\chi(M) := \sum_{i=0}^{d}(-1)^i n_i = \sum_{i=0}^{d}(-1)^i \beta_i$$

More generally, β_n is the number of n dimension holes of the topological space considered.

Implementation: We give two ways compute the homology groups (and possibly their generators).

1. By reducing the incidence matrices (i.e the matrices of boundary operators) [7] under the *Smith normal form*. The Smith normal form of an integer matrix is the equivalent matrix which is diagonal and the non zero coefficients $d_1, d_2, \cdots, d_m$ verify $d_1 \mid d_2 \mid \cdots\ d_m$ as in structure theroem of finitely generated abelian groups. This reduction gives the Betti numbers, the torsion coefficients, a basis of n-cycles and n-boundaries but not a basis of homology groups.
From this normal form, we can compute the *Smith-Agoston normal form* which is similar to the previous one. With this normal form, we have the generators of homology groups.

2. Using the *Mayer-Vietoris exact sequence* [10]. The main idea of this method is to compute the homology by decomposing the space in two smaller spaces. It is interesting only if the homology of the two subspaces and their intersection is easier to compute than for the initial one.

As we said in introduction, in data processing we consider algebraic data structures to study the topology. These data structures are often topological invariants and then gives topological equivalences which are less stronger than homemomorphism. We summarize these equivalences and the relations between themselves:

Isotopy $\Rightarrow$ Homeomorphism $\Rightarrow$ Homotopy $\Rightarrow$ Homology.

Definition 6: An *ambiant isotopy* between two surfaces S and S' in $\mathbb{R}^3$, is a continuous map

$$\Gamma : \mathbb{R}^3 \times [0,1] \longrightarrow \mathbb{R}^3$$

such that $\Gamma(.,t)$ is an homeomorphism from $\mathbb{R}^3$ to itself for each $t \in [0,1]$, and $\Gamma(S,1) = S'$.

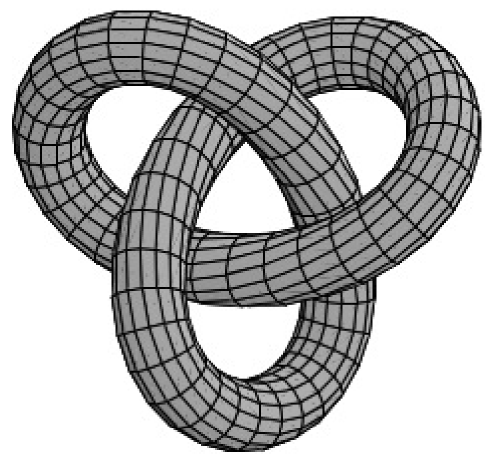

Fig. 4: A geometric object which is homeomorphic to the torus but not isotopic

In $\mathbb{R}^3$, the cylinder and the circle are homotopic (the circle is a *deformation retract* of the cylinder) but not homeomorphic.

III. Morse theory

We are focused here in differentiable manifolds, their topology is studied using smooth functions defined on them. The main idea of Morse theory it is enough to study the topology of the manifold at the neighborhood of each critical point.
Let M be a differentiable n-manifold and $h : M \longrightarrow \mathbb{R}$ a smooth function. We note: $M_h^a := h^{-1}(]-\infty, a])$.

A. Topology of manifolds [3]

Definition 7: We say that $h : M \longrightarrow \mathbb{R}$ is a *Morse function* if all its critical points are non-degenerate.

There are three types of critical points non-degenerate: minimums, saddle points, maximums.

Definition 8 (Morse Index): The *Morse index* of p for h is defined by one of the equivalent following assertions:
(i) the number of minus signs in the the development of f using the Morse lemma
(ii) the number of negative eigenvalues of the Hessian matrix of f at the point p
(iii) the number of independant directions where f decreases from p.

Examples: If M is a 2-manifold in $\mathbb{R}^3$ and $p \in M$ a critical point non-degenerate of h, then:

$$i_h(p) = \begin{cases} 0 & \text{if } p \text{ is a minimun} \\ 1 & \text{if } p \text{ is a saddle point} \\ 2 & \text{if } p \text{ is a maximum} \end{cases} .$$

Theorem 4: Let $a < b$, we assume that $h^{-1}([a,b])$ is non empty, compact (it is always the case if M is compact) and does not contain critical points.
Then, M^a and M^b are diffeomorphic.

This theorem justify the fact that it is sufficient to study the topology of the manifold at the neighborhood of each critical point.

Corollary 5 (Reeb theorem): Assume that M is compact and there exists a Morse fonction defined on M with only two critical points. Then, M is homeomorphic to a sphere.

We now give a theorem which gives the evolution of the topology when you cross a critical point.

Theorem 6: We assume that $p \in M$ is a critical point of h and that: $\exists\, \epsilon > 0,\ h^{-1}([\alpha - \epsilon, \alpha + \epsilon])$ is compact and contains no other critical point that p. We note: $\alpha = h(p)$.
Then, $M_h^{p+\epsilon}$ is homotopic to $M_h^{p-\epsilon}$ which is added a $(n - i_h(p))$-cell (which is in fact the unstable manifold $W_h^u(p)$ (see definition 9)).

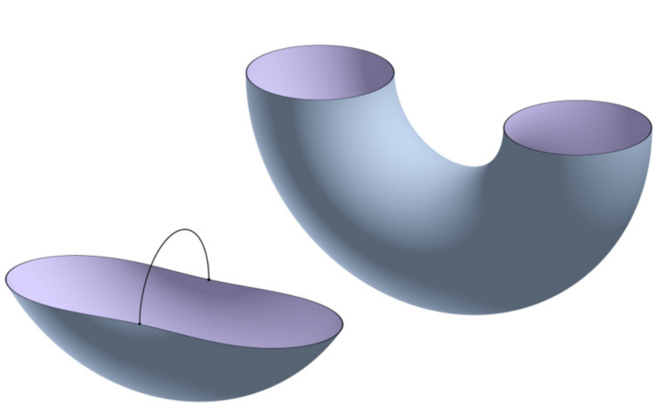

Fig. 5: changement of topology at the neighborhood of the first saddle point

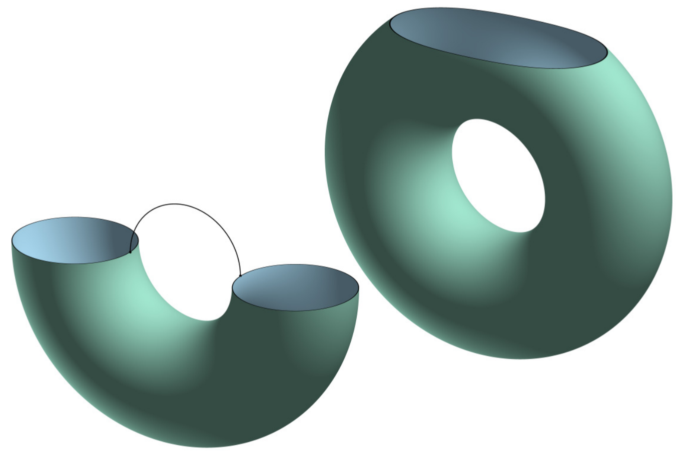

Fig. 6: changement of topology at the neighborhood of the second saddle point

Proposition 7: We have the two following relations:
1. [Morse inequality].

$$\forall\, k \in \mathbb{N},\ \beta_k(M) \leq \mu_k(h)$$

where: $\mu_k(h) := |\{x \in M \mid i_h(x) = k\}|$
2.

$$\chi(M) = \sum_{i=0}^{d} (-1)^i \beta_i(M) = \sum_{i=0}^{d} (-1)^i \mu_i(h).$$

Examples: 1. For the sphere $S^2 \subset \mathbb{R}^3$ provided with the height function h, we have:

$$\mu_k(h) = \begin{cases} 2 & \text{if } k \in \{0,2\} \\ 0 & \text{otherwise} \end{cases}$$

$$\beta_k(S^2) = \begin{cases} 1 & \text{if } k \in \{0,2\} \\ 0 & \text{otherwise} \end{cases}$$

and therefore $\chi(S^2) = 2$.
2. We consider the torus $T^2 \subset \mathbb{R}^3$ provided with the height function h, we have: $\chi(T^2) = 0$.
Indeed, we have:

$$\beta_k(T^2) = \begin{cases} 1 & \text{if } k \in \{0,2\} \\ 2 & \text{if } k = 1 \\ 0 & \text{otherwise} \end{cases}$$

and:

$$\mu_k(h) = \begin{cases} 1 & \text{if } k \in \{0,2\} \\ 2 & \text{if } k = 1 \\ 0 & \text{otherwise} \end{cases} .$$

B. Morse-Smale complex [12] [13]

We are focused in the following differential problem:

$$(E) : \begin{cases} x'(t) = \nabla h(x(t)) \\ x(0) = x_0 \in M \end{cases}$$

it admit an unique maximal solution (Cauchy-Lipschitz theroem on a manifold). If M is compact, then the maximal solution of (E) is defined on $\mathbb{R}$.

Notation: γ_p the integral curve through the point $p \in M$.

Proposition 8: Let x be an integral curve for h, we have the following properties:
1. The integral curves are strictly increasing (according to h) or constants and form a partition of M.
2. If the limits $\lim_{t\to+\infty} x(t)$ and $\lim_{t\to-\infty} x(t)$ exist (it is always the case is M is compact), then they are critical points.
3. The integral curves are orthogonal to regular level sets.

Definition 9: Let $p \in M$, we define the two following sets:
1. $W_h^s(p) := \{x \in M \mid lim_{t\to+\infty}\ \gamma_x(t) = p\}$: it is the *stable manifold of* p.
2. $W_h^u(p) := \{x \in M \mid lim_{t\to-\infty}\ \gamma_x(t) = p\}$: it is the *unstable manifold of* p.

Proposition 9: We have the following properties:
1. $W_{-h}^s(p) = W_h^u(p)$; $W_{-h}^u(p) = W_h^s(p)$.
2. M is covered by the stable and unstable manifolds which are open sets of $\mathbb{R}^d$ (with d which depend on the stable/unstable manifold).

Theorem 10: Let p be a critical point, then the sets $W_h^s(p)$ and $W_h^u(p)$ are differentiable manifolds.
Moreover: $\dim W_h^s(p) = i_h(p)$
and therefore $\dim W_h^u(p) = n - i_h(p)$.

Particular case: If $M \subset \mathbb{R}^3$ is a 2-manifold, $p \in M$ a saddle point ($i_h(p) = 1$), then we have: $W_h^s(p) \setminus \{p\}$ (resp. $W_h^u(p) \setminus \{p\}$) is the union of two curves coverging to (resp. diverging from) p. These curves are called the stable (resp. unstable) *separatrices* of the saddle point.

Definition 10: $h : M \longrightarrow \mathbb{R}$ is a *Morse-Smale* function if:
1. h is a Morse function
2. the stable and unstable manifolds intersect only transversally, that is to say:
$\forall\, x \in W_h^s(p) \cap W_h^u(q),\ T_xM = T_xW_h^s(p) + T_xW_h^u(q)$.

Important remark: If h is a Morse-Smale function defined on $M \subset \mathbb{R}^3$ a 2-manifold, then there is no integral curve which connects two saddle points.

Definition 11: We assume that h is a Morse-Smale function. The *Morse-Smale complex* of M provided with h is the subdivision of M formed by the connected components of the intersections $W_h^s(p) \cap W_h^u(q)$ when p and q range over all critical points of h.

More precisely, the Morse-Smale complex of a 2-manifold in $\mathbb{R}^3$ can be obtained as follow:
1. the vertices are the critical points
2. the edges are integral lines connecting a minimun (resp. maximum) at a saddle point
3. the regions (i.e 2-cells) are the integral lines connecting a minimum at a maximum.

Lemma 11 (quadrangle lemma): Each region of the Morse-Smale complex of a 2-manifold in $\mathbb{R}^3$ is a quadrangle. Futhermore, the Morse indices of the vertices of these quadrangles are $0, 1, 2, 1$ in this order.

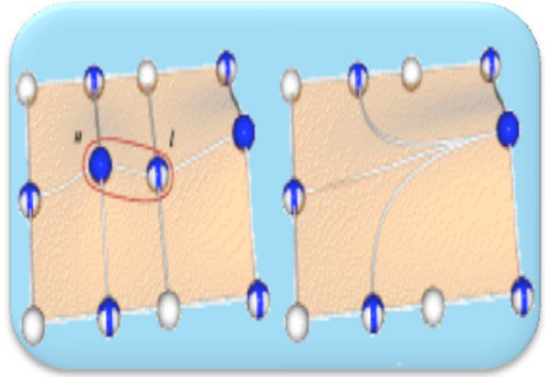

Fig. 7: Morse-Smale complex

The blue points are the maximiums (index 2), the white are the minimums (index 0) and the blue and white are the saddle point (index 1).

Applications:
· in molecular form analysis [11]
· in topological data analysis [15]
· in vector fields analysis [16] [17].

For the computation of Morse-Smale complexes, Edelsbrunner *et al.* propose a method for piecewise linear 2-manifolds [19] (resp. 3-manifolds [20]) using the notion of quasi Morse-Smale complexes.

IV. Cylindrical Algebraic Decomposition

The main objective of this algorithm is to create a cylindrical decomposition which partitions the set S into connected subsets compatible with the zeros of the polynomials. This means that on each subset of the CAD, each of the polynomials either vanishes everywhere or nowhere.

Definition 12 (Cylindrical decomposition): A decomposition of $\mathbb{R}^n$ into finitely many connected regions is *cylindrical* if for any two regions A and B of the decomposition and any k, $1 \leq k \leq n$, the projections of A and B onto $\mathbb{R}^k$ are either identical or disjoint. A cylindrical decomposition D of $\mathbb{R}^n$ induces cylindrical decompositions of $\mathbb{R}^k$ for every $k \leq n$.

This definition leads us to construct a cylindical (algebraic) decomposition of $\mathbb{R}^{n+1}$ from one of $\mathbb{R}^n$.

Definition 13 (Cylindrical Algebraic Decomposition): A cylindrical decomposition of $\mathbb{R}^n$ into semi-algebraic sets is a *cylindrical algebraic decomposition*.

More formally, we can define a cylindrical decomposition by means of a recurrent definition. A cylindrical decomposition of $\mathbb{R}^n$ is a decomposition of $\mathbb{R}^n$ in cells $(C_i)_i$ for which:

- $n = 1$: A cylindrical decomposition of $\mathbb{R}$ is a subdivision $a_1 < \ldots < a_l$, with $a_i \in \mathbb{R}$ for all $1 \leq i \leq l$.
- $n > 1$: A cylindrical decomposition of $\mathbb{R}^n$ is given by a cylindric decomposition of $\mathbb{R}^{n-1}$ such that for each cell C of the decomposition of $\mathbb{R}^{n-1}$ there exist $l(C)$ semi-algebraic functions

$$\varphi_{C,1} < \ldots < \varphi_{C,l(C)} : D \to \mathbb{R}.$$

Input: A finite family $\mathcal{P}_n$ of polynomials on $\mathbb{R}[X_1, \ldots, X_n]$.
Output: A cylindrical algebraic decomposition adapted to $\mathcal{P}_n$.

1) **if** $n = 1$ **then**
2) Return a cylindrical decomposition D_1 of family $\mathcal{P}_1$.
3) **else**
4) Construct a family of polynomials $\mathcal{P}_{n-1}$ on $\mathbb{R}[X_1, \ldots, X_{n-1}]$ such that a decomposition adapted to $\mathcal{P}_{n-1}$ is the base of a decomposition adapted to $\mathcal{P}_n$.
5) $D_{n-1} \leftarrow$ Collins($\mathcal{P}_{n-1}$).
6) For every cell C in decomposition D_{n-1}, compute the decomposition of the cylinder under C induced by polynomials $\mathcal{P}_n$.
7) Add all those cells to D_n and return D_n.
8) **end if**

Fig. 8: Algorithm for the Cylindrical Algebraic Decomposition.

The $(n-1)$-cells of the cylindrical decomposition of $\mathbb{R}^n$ are the graphs of the functions $\varphi_{C,i}$

$$\{(x, \varphi_{C,i}(x));\ x \in D\}, \quad 0 < i \leq l(C).$$

The n-cells are the sets $]\varphi_{C,i}, \varphi_{C,i+1}[\ =$

$$\{(x, y) \in D \times \mathbb{R} / \varphi_{C,i}(x) < y < \varphi_{C,i+1}(x)\},$$
$$0 < i \leq l(C),$$

with $\varphi_{C,0} = -\infty$ and $\varphi_{C,l(C)+1} = +\infty$.

For $0 \leq i \leq n$, an i-cell is a subset of $\mathbb{R}^n$ which is homeomorphic to $\mathbb{R}^i$, and every element of a cylindrical decomposition is an i-cell for some i.

Given a finite family $\mathcal{P}_n$ of polynomials on $\mathbb{R}[X_1, \ldots, X_n]$, Collins algorithm constructs a cylindrical algebraic decomposition of $\mathbb{R}^n$ such that all polynomials have constant sign at every cell. Such a decomposition is called adapted to $\mathcal{P}_n$.

Collins algorithm is a recursive algorithm. The main concept is as follows: from a finite family $\mathcal{P}_n$ of polynomials on $\mathbb{R}[X_1, \ldots, X_n]$, create a family $\mathcal{P}_{n-1}$ of polynomials on $\mathbb{R}[X_1, \ldots, X_{n-1}]$ such that a decomposition adapted to $\mathcal{P}_{n-1}$ is the base of a decomposition adapted to $\mathcal{P}_n$. Algorithm 8 presents this method.

For the case $n = 1$, the decomposition is made by the set of real roots of the polynomials of $\mathcal{P}_1$. The 0-cells will be the roots, and the 1-cells will be the intervals defined by two subsequent roots.

For the case $n = 2$, we are going to present a limited version for $\mathbb{R}^2$. The general case is a little more complicated but general concepts remain the same.

Given is a family $\mathcal{P}_2$ of polynomials with two variables. We will find a decomposition D_1 adapted to $\mathcal{P}_1$. The constraints that the cells in the decomposition D_1 must hold are:

1) For every cell C in D_1, all polynomials in $\mathcal{P}_2$ must have a constant number of roots as polynomials of variable x_2.
2) For every open cell C in D_1, the curves defined by the roots of the elements in $\mathcal{P}_2$ never intersect.

We analyze the conditions for those two constraints to hold. For the first one, there are two possibilities:

- A real root becomes complex. Then, singletons $\{x_1\}$ verifying
$$\exists\ x_2 \in \mathbb{R}, P(x_1, x_2) = 0 = \frac{\partial P}{\partial x_2}(x_1, x_2) \quad (1)$$
must be in D_1.
- A real root becomes infinite. Then, if $P(x_1, x_2)$ can be written as $a_n(x_1)\ x_2^n + \ldots + a_0(x_1) = 0$ with $a_i \in \mathbb{R}[X_1]$, the singletons $\{x_1\}$ verifying $a_n(x_1) = 0$ must be in D_1.

For the second one, there are also two possibilities:

- Both curves are roots of the same polynomial. Then $P(x_1, x_2)$ has a multiple root and equation 1 must be fulfilled.
- Both curves are roots of two different polynomials P and Q. Then, singletons $\{x_1\}$ verifying
$$\exists\ x_2 \in \mathbb{R}, P(x_1, x_2) = 0 = Q(x_1, x_2) \quad (2)$$
must be in D_1.

In order to find the univariate polynomials on x_1 which will belong to D_1 such that their zeros contain the previously enumerated elements of D_1, we have to add $a_n(x_1)$ to $\mathcal{P}_1$ for the second case, and for the other cases $\mathcal{P}_1$ must contain polynomials which vanishes whenever equations 1 and 2 hold.

V. C.I.A and H.I.A algorithms [4]

In this section, we consider topological spaces defined by CSG (constructive solid geometry) trees. More precisely, a such space E is defined as follow:

$$E := \bigcup_{i=1}^{s} \bigcap_{j=1}^{r_i} \{x \in \mathbb{R}^n \mid f_{i,j}(x) \diamond_{i,j} 0\}\ (\sharp)$$

where: $f_{i,j}$ are $\mathcal{C}^1$-functions and
$\diamond_{i,j} \in \{=\ ;\ \leq\ ;\ \geq\}$.

A. Connected components: C.I.A algorithm [5]

This algorithm computes the path-connected components of a topological space defined as in ($\sharp$). For that, it covers the space E by a collection of boxes $(p_i)_{i\in I}$ such that:

$$\forall\ i \in I,\ E \cap p_i \text{ is star-shaped (and therefore path-connected).}$$

We start by giving a sufficient condition for a point to be a star.

Proposition 12: Let $v \in E$, is the following system:

$$f(x) = 0,\ d_x f(x - v) \leq 0,\ x \in E$$

is inconsistent (i.e: has no solutions), then v is a star for E.

Definition 14: A *star-spangled graph* of E, noted G_E, is a relation $\mathcal{R}$ on a tiling $\mathcal{P}$ where:

- $\mathcal{P} = (p_i)_{i\in I}$ is a tiling,

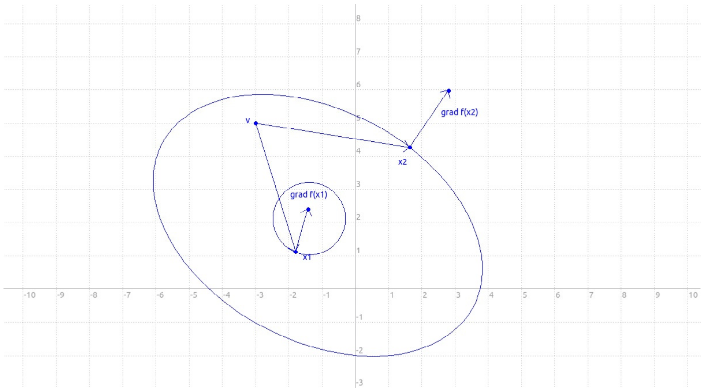

Fig. 9: Star test: we have $d_{x_1}f(x_1-v) > 0$ and $d_{x_2}f(x_2-v) \leq 0$, then v is not a star.

- $\mathcal{R}$ is the reflexive, symetric relation defined by:
$$p\,\mathcal{R}\,q \Longleftrightarrow E \cap p \cap q \neq \emptyset$$
- $E \subset \cup_{i \in I}\ p_i$.

Proposition 13: With a such decomposition, we have the following fundamental result:
$|\{$ path-connected components of $E\}| = |\{$ connected components of the graph $G_E\}|$.
In particular, E is path-connected if and only if G_E is connected.

C.I.A algorithm

Notations:
$\mathcal{P}_\star$ contains boxes p such that $E \cap p$ is star-shaped.
$\mathcal{P}_{out}$ contains boxes p such that $E \cap p$ is empty.
$\mathcal{P}_\Delta$ contains boxes such that nothing is known about them.

This algorithm decomposes itself in three algorithms:

- Star-shaped (E, p) which determines if $E \cap p$ is star-shaped or not (thanks to proposition 15)
- Build-graph-interval $(E,\mathcal{P})$ which computes the star-spangled graph of E provided with the tiling $\mathcal{P}$.
- C.I.A (E, X_0) which computes the path-connected components of E.

Star-shaped (E, p):
Inputs: $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ a $\mathcal{C}^1$-function, p a box of $\mathbb{R}^n$.

1. If $f(p) \subset \mathbb{R}_+^\star$ then we return "$E \cap p = \emptyset$"
2. else for all vertex v_p of p do
3. if $\{x \in p,\ f(x) = 0,\ f(v_p) \leq 0,\ d_x f(x - v_p) \leq 0\}$ is inconsistent then we return "$E \cap p$ is star-shaped"
4. end if
5. end for
6. return "fail"
7. end if.

Build-graph-interval $(E, \mathcal{P})$:
Inputs: $E \subset \mathbb{R}^n$, $\mathcal{P}$ a tiling

Outputs: an interval graph $[\underline{g}, \overline{g}]$

1. $\underline{g} := \emptyset$, $\overline{g} := \emptyset$
2. for all $(p_i, p_j) \in \mathcal{P} \times \mathcal{P}$ do
3. if $E \cap p_i \cap p_j \neq \emptyset$
4. then if a vertex of $p_i \cap p_j$ is in E then we add (p_i, p_j) to $\underline{g}$ and to $\overline{g}$
5. else we add (p_i, p_j) to $\overline{g}$
6. end if
7. end if
8. end for
9. return the interval graph $[\underline{g}, \overline{g}]$.

C.I.A (E, X_0) : path-connected components using interval analysis :
Inputs : $E \subset \mathbb{R}^n$, X_0 a box of $\mathbb{R}^n$ which contains E.

1. $\mathcal{P}_\star := \emptyset$, $\mathcal{P}_\Delta := \{X_0\}$, $\mathcal{P}_{out} := \emptyset$
2. While $\mathcal{P}_\Delta \neq \emptyset$ do
pop the last element p of $\mathcal{P}_\Delta$
3. if $E \cap p = \emptyset$ then do $\mathcal{P}_{out} \leftarrow \mathcal{P}_{out} \cup \{p\}$ and go to step 2
4. else if $E \cap p$ is star-shaped and Build-graph-interval$(E, P_\star \cup \{p\}) = [\underline{g}, \overline{g}]$ is punctual (i.e $\underline{g} = \overline{g}$) then push p in $\mathcal{P}_\star$ and go to step 2
5. else subdivide the box p in two boxes p_1 and p_2, push $p1$ in $\mathcal{P}_\Delta$, push $p2$ in $\mathcal{P}_\Delta$ and go to step 2
6. end if
7. end while
8. $[\underline{g}, \overline{g}] :=$ Build-graph-interval $(E, \mathcal{P}_\star)$
9. $n \leftarrow |\{$ connected components of $\underline{g}\}|$
10. return "E has n path-connected components".

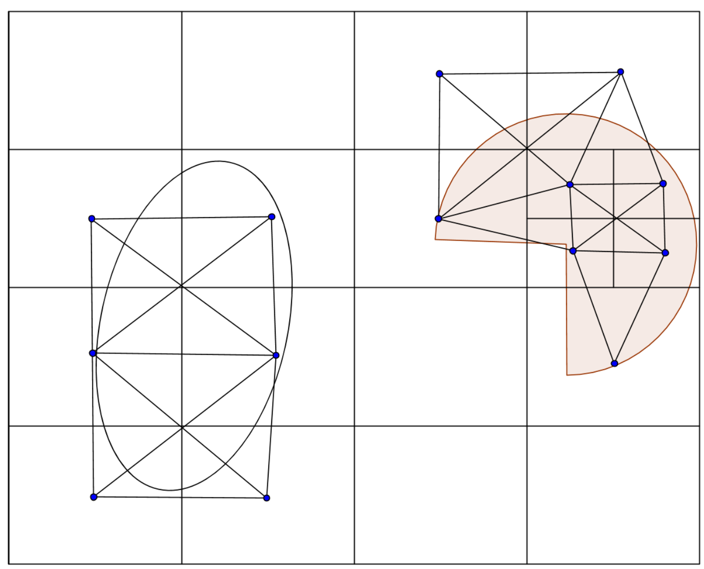

Fig. 10: Path-connected components: C.I.A algorithm

B. Homotopy type: H.I.A algorithm [6]

This algorithm computes a simplicial complex which is homotopic to E.

Definition 15: We say that a space is *contractible* if it is homotopic to a point.

Example: The star-shaped spaces are contractibles.

Definition 16: We say that $\{p_i\}_{i\in I}$ is a *compact contractible covering* of E if:
(i) I is finite
(ii) $\forall\ i \in I,\ p_i$ is compact
(iii) $\forall\ J \subset I,\ p_J := \cap_{i\in J}\, p_i\ \cap\ E$ is contractible or empty.

Definition 17: 1. An *abstract simplicial complex* $\mathcal{K}$ is a subset of $\mathcal{P}(\{a^0, a^1, .., a^n\})$ such that:

$$\forall \sigma \in \mathcal{K}, \forall \tau \subset \sigma, \tau \in \mathcal{K}$$

The a^i are called *abstract vertices* ans subsets of $\{a^{i_0}, ..., a^{i_s}\}$ $(s \in \mathbb{N}^\star)$ the *abstract simplices*.
2. The *dimension* of an abstract simplex $\{a^{i_0}, ..., a^{i_s}\}$ is s. The dimension of an abstract simplicial complex is the maximal dimension of its abstract simplices.

Example : $\mathcal{K}_0 := \{\{a^0\}, \{a^1\}, \{a^2\}, \{a^0, a^1\}, \{a^0, a^2\}, \{a^1, a^2\}, \{a^0, a^1, a^2\}, \{a^3\}, \{a^4\}, \{a^3, a^4\}\}$
$dim\ \mathcal{K}_0 = dim\ \{a^0, a^1, a^2\} = 2.$

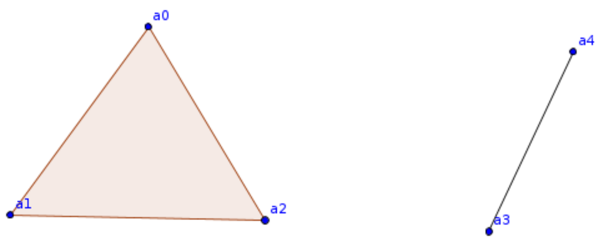

Fig. 11: The abstract simplicial complex generated by $\mathcal{K}_0$

Definition 18: Let σ_1,..., $\sigma_m \in \mathcal{P}(\{a^0, ..., a^n\})$. We note $\sigma_1 + ... + \sigma_m$ the abstract simplicial complex generated by $(\sigma_i)_{i\in[|1,m|]}$ defined by:

$$\sigma_1 + ... + \sigma_m := \bigcup_{i=1}^{m} \mathcal{P}(\sigma_i)$$

Example: $\mathcal{K}_0 = \{a^0, a^1, a^2\} + \{a^3, a^4\}$.

Definition 19: Let $\mathcal{K}$ an abstract simplicial complex and x an abstract node such that $\{x\} \notin \mathcal{K}$.
We note $\mathcal{C}(x, \mathcal{K})$ the abstract simplicial complex generated by x and $\mathcal{K}$, that is to say we have the following formula:

$$\mathcal{C}(x, \mathcal{K}) := \mathcal{K} \cup \bigcup_{s\in\mathcal{K}} \{x\} \cup s$$

Notation: Si $\mathcal{K} = \sigma_1 + ... + \sigma_m$, then we note:

$$\mathcal{C}(x, \mathcal{K}) = x * (\sigma_1 + ... + \sigma_m) := x * \sigma_1 + ... + x * \sigma_m$$

where: $x * \sigma := \{x\} \cup \sigma$.

Definition 20: Let $\{p_i\}_{i\in I}$ a compact contractible covering of E.
We note: $\mathcal{J} := \{J \subset I \mid p_J \neq \emptyset\}$.
We say that an abstract simplicial complex $\mathcal{K}(E)$ is *adapted* to $\{p_i\}_{i\in I}$ if it is the smallest simplicial complex such that:

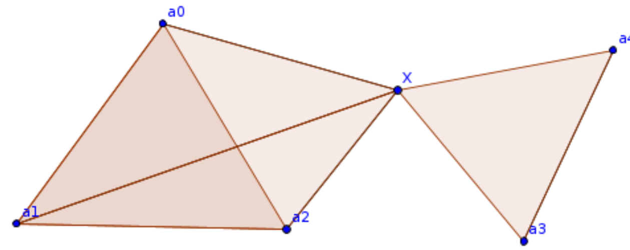

Fig. 12: The abstract simplicial complex $\mathcal{C}(x, \mathcal{K}_{\prime})$

- $\forall\ J \in \mathcal{J}$, an abstract vertex (a^J) is in $\mathcal{K}(E)$
- $\forall\ J \in \mathcal{J}$, an abstract simplicial complex $\mathcal{K}_J$ defined by:

$$\mathcal{K}_J := a^J * \left(\sum_{J'\in\mathcal{J}|p_{J'}\subset p_J} \mathcal{K}_{J'} \right)$$

is an abstract simplicial sub-complex of $\mathcal{K}(E)$.

Theorem 14: (Nerve theorem) If $\{p_i\}_{i\in I}$ is a covering of E and if $\mathcal{K}(E)$ is an abstract simplicial complex adapted to $\{p_i\}_{i\in I}$, then E and $\mathcal{K}(E)$ have the same homotopy type.

H.I.A algorithm:

Notations:
$\mathcal{P}_\star$ the tiling such that: $\forall\ \{p_j\}_{j\in J} \subset \mathcal{P}_\star,\ \cap_{j\in J} E \cap p_j$ is contractible or empty.
$\mathcal{P}_\triangle$: nothing is known about its boxes.

This algorithm decompose itself in two algorithms:

Nerve $(E, \mathcal{P} = \{p_i\}_{i\in I})$
Inputs: a set E is a covering $\{p_i\}_{i\in I}$ (p_i are boxes).
Outputs: un complexe simplicial abstrait $\mathcal{K}(E)$ which is adapted to $\{p_i\}_{i\in I}$.

1. $\mathcal{K}(E) \leftarrow \emptyset$; $\mathcal{J} \leftarrow \emptyset$
2. for all $J \subset I$ do
3. if E_J is contractible then $\mathcal{J} \leftarrow \mathcal{J} \cup \{I\}$
4. end if
5. end for
6. $\mathcal{K}(E) \leftarrow \sum_{i\in I'} Cone(\{i\})$

where: $I' := \{i \in I | \{i\} \in \mathcal{J}\}$
and $Cone(J)$ is recursively defined for $J \in \mathcal{J}$ by the following formula:

$$Cone(J) := a^J * \left(\sum_{J'\in\mathcal{J}|E_{J'}\subset E_J} Cone(J') \right)$$

.

H.I.A (E, X_0): homotopy type via interval analysis
Inputs: $E \subset \mathbb{R}^n$, X_0 a box of $\mathbb{R}^n$ which contains E.
Outputs: an abstract simplicial complex $\mathcal{K}(E)$ which is

homotopic to E.

1. $\mathcal{P}_\star := \emptyset$; $\mathcal{P}_\Delta := \{X_0\}$
2. while $\mathcal{P}_\Delta \neq \emptyset$ faire
3. pop the last element p of $\mathcal{P}_\Delta$
4. if $\forall \{p_j\}_{j\in J} \subset \mathcal{P}_\star \cup \{p\}$, $\cap_{j\in J} E \cap p_j$ is contractible or empty then push p in $\mathcal{P}_\star$
5. else subdivide p in two boxes p_1 and p_2, push $p1$ in $\mathcal{P}_\Delta$, empiler $p2$ in $\mathcal{P}_\Delta$
6. end if
7. end while
8. $\mathcal{K}(E) \leftarrow$ Nerve $(E, P_\star)$.

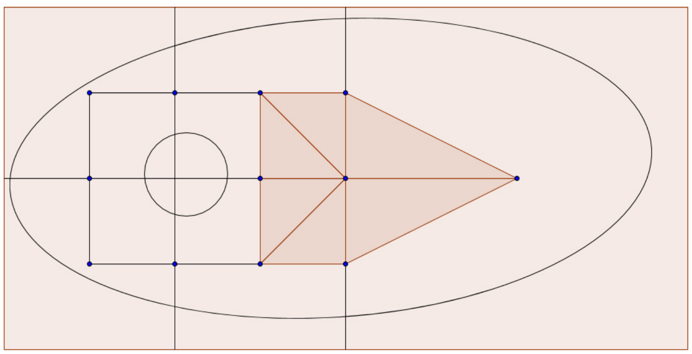

Fig. 13: homotopy type, H.I.A algorithm

Application: An important application of these two algorithms is the path planning [9], especially in robotics where the CIA algorithm used to determine feasible positions from a given configuration of the robot.

Now we present how Varadhan & Manocha apply the technique in HIA to compute a roadmap in motion planning.

C. Varadhan & Manocha's method [18]

Varadhan & Manocha's method is based in the same idea as Delanoue *et al.*'s, and present a very similar method to compute the star-shaped decomposition of the original set. So we will consider that the method to create so a decomposition is irrelevant.

Based on star-shapedness the intra-region connectivity is captured. In order to achieve also inter-region connectivity the concept of *connector* is introduced. The presented approach to compute the roadmap is:

1) Compute a star-shaped decomposition Σ of the free space.
2) For every pair of adjacents regions in Σ, compute a point c (*connector*) on their common boundary.
3) Construct a star-shaped roadmap $\mathcal{R}$ as the undirected graph (V, E). Denoting by S and C the set of star-points and connectors respectively, $V = S \cup C$. Each connector c connects two star-points s_1, s_2 of two adjacent regions. Let $Stars(c)$ denote $\{s_1, s_2\}$. Then,

$$E = \{(c, s) : c \in C, s \in Stars(c)\}$$

Theorem 15: Let $\mathcal{F} \subset \mathbb{R}^n$ and $p, q \in \mathcal{F}$ two points in it. Let $\mathcal{R}$ be a star-shaped roadmap for $\mathcal{F}$ defined as before. Then p and q are connected in $\mathcal{F}$ if and only if $\exists\ p^*, q^* \in \mathcal{F}$ such that p and p^* are connected in $\mathcal{F}$, p^* and q^* are connected in $\mathcal{R}$ and q and q^* are connected in $\mathcal{F}$.

This theorem sets the basis to find a collision-free path between two points in $\mathcal{F}$.

VI. CONCLUSION

We presented in this paper a non-exhaustive list of data structures and algorithms describing topology and/or geometry. As regarding the computation of homology groups, we need to know how to compute a simplicial complex on the topological space (ie bring it to a Δ-complex structure). Another method is to compute a simplicial complex which is homotopic to the geometric object, for example using the HIA algorithm.
The CIA and HIA algorithms do not apply as such to sets defined by projection. It is possible to generalize the two algorithms to solve this problem: a new representation of geometric sets is used for that [14]. We also would like to know if you can replace homotopic by isotopic in the HIA algorithm.

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