INCIDENCE CONSTRAINTS: A COMBINATORIAL APPROACH

Dominique MICHELUCCI

LE2I, Université de Bourgogne, Dijon[*]

Pascal SCHRECK

LSIIT, Université Louis Pasteur, Strasbourg[†]

The simplest geometric constraints are incidences between points and lines in the projective plane. Surprisingly, this problem is universal, in the sense that all algebraic systems reduce to such geometric constraints. Detecting dependences between these geometric constraints is NP-complete. New methods to prove incidence theorems are proposed; they use strictly no computer algebra but only combinatorial arguments.

Keywords: Projective geometry, incidence constraint, combinatorial proof, hexamys, compatible matroid

1. Overview

Geometric solvers[2] are today a key component of all geometric modelers used in CAD-CAM (Computer Aided Design, Computer-Aided Manufacturing). They enable to define the relative location of geometric entities (points, lines, planes, curves, surfaces) with geometric constraints: distance or angle between geometric elements, tangency or incidence relations. Typically, a preprocessing step detects over- and under-constrained systems, and decomposes well-constrained systems into irreducible components faster and easier to solve. Then some numerical solver is often used, since the complexity of the irreducible well-constrained systems prevents using computer algebra[3].

The methods for decomposing systems of constraints are typically graph-based. They detect only the structural dependences between constraints, they do not capture more subtle dependences due to geometric theorems; dependent (redundant or contradictory) systems mislead the numerical solver and prevent resolution.

[*]Laboratoire d'Electronique, Informatique et Image - Université de Bourgogne, 21000 Dijon, France, Dominique.Michelucci@u-bourgogne.fr
[†]Laboratoire des Sciences de l'Image, de l'Informatique et de la Télédétection, UMR CNRS-Université Louis Pasteur 7005, 67084 Strasbourg, France, Pascal.Schreck@dpt-info.u-strasbg.fr

2   *D. Michelucci and P. Schreck*

These dependences do not disappear when users modify their sketch or the values of length or angle parameters; this phenomenon (which is terribly frustrating for users) suggests that the dependence is due to a theorem of projective geometry, such as Desargues, Pappus or Pascal (2).

This paper studies the possibility and the difficulty of detecting such dependences with combinatorial methods, in the wake of the Montréal Structural Topology[17] and of the Rigidity Theory[15]. Due to the difficulty of the problem, we focus on the detection of dependences in incidence constraints between flats (points, lines, planes), in particular in the projective plane[5,7,18]. The seemingly trivial problem is as follows: a user specifies that some subsets of points are collinear, and that others are not; is the specification realizable in some projective plane?

The rest of this paper is structured as follows. Section 2 shows that considering only incidence constraints between points and lines in the projective plane is not an academic toy problem since all algebraic systems of arbitrary degree in a field $\mathbb{K}$ can be reduced to an equivalent system of point-line incidences in $\mathbb{P}(\mathbb{K})$, the projective plane with coordinates in $\mathbb{K}$ (2.3). Section 3 describes the *compatible matroid problem*: if a system of incidences is realizable, there is a matroid compatible with the specified incidences; section 3 proves that in 2D and beyond, this problem is NP-complete. Conversely, some compatible matroids are not realizable: they contain forbidden configurations, contradicting theorems of projective geometry. Finding a compatible matroid free of forbidden configurations is the *compatible configuration problem*. Section 4 and 5 describe two provers: Hexamys and Pappus, which solve the compatible configuration problem. Both are half deterministic. It is due to the non monotony of the compatible configuration problem. Section 6 conjectures that combinatorial provers can reach full determinism: it argues that the compatible matroid problem is monotone, and Pappus theorem is captured by some rank 10 matroids. Section 7 concludes.

2. Fundamentals of projective geometry

2.1. *Basic facts*

Up to now, in the field of Geometric Constraints Solving, researchers have mainly studied rigid systems of constraints in order to specify CAD-CAM objects. These constraints concern metric properties of the objects to be designed. Consequently, the subtended framework was Euclidean geometry associated with the group of isometries, or conform geometry with the similarity group[14].

We put here the focus on incidence constraints, and then, on a simplified framework where only points, lines and incidence relationship are considered. This framework corresponds to *incidence geometry*. For the sake of simplicity, we consider *projective geometry*: two distinct coplanar lines always meet in one point. Projective geometry is associated with the group of homographies.

Incidence constraints are the simplest geometric constraints, but they can lead to arbitrarily difficult configurations. Fig. 1 illustrates a configuration which is not
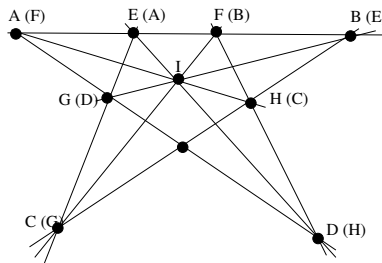
Fig. 1. An irrational configuration. Lines are $ABEF$, ADG, AHI,BCH, BIG, DFH, CFI, DEI (and all pairs of points). In parenthesis, another automorphic solution.
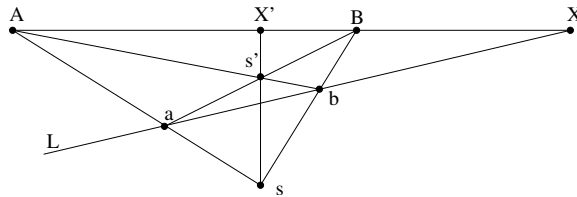

Fig. 2. For given aligned points A, B, X, point X' does not depend on L nor s.

realizable with rational coordinates in the real projective plane. Indeed, a regular pentagon –or, more precisely, the image of a regular pentagon by a homography– is needed. Other small systems of incidence constraints have been studied[5,13,9].

2.2. *Classical 2D Theorems*

Let us recall some classical properties of the real projective plane $\mathbb{P}$ and related to incidence relationship. They all share the same combinatorial flavor which becomes more explicit in further sections.

(Harmonic conjugate) Let A, B, X be 3 distinct aligned points. Let L be any line through X, s any point outside L and ABX. Then point X' defined by the construction: $a = sA \cap L$, $b = sB \cap L$, $s' = aB \cap Ab$, $X' = ss' \cap AB$, depends neither on L nor on s. See Fig. 2.

(Pappus theorem) If p_1, p_2, p_3 are three distinct aligned points of $\mathbb{P}$, and if q_1, q_2, q_3 are three distinct aligned points, then the three intersection points $p_1q_2 \cap p_2q_1$, $p_1q_3 \cap p_3q_1$ and $p_2q_3 \cap p_3q_2$ are aligned as well.

(Dual of Pappus) If l_1, l_2, l_3 are three distinct concurrent lines of $\mathbb{P}$, and if m_1, m_2, m_3 are three distinct concurrent lines, then the three joining lines $(l_1 \cap m_2, l_2 \cap m_1), (l_1 \cap m_3, l_3 \cap m_1), (l_2 \cap m_3, l_3 \cap m_2)$ are concurrent as well.

(Desargues theorem) If two triangles abc and ABC are perspective (that is aA, bB, cC concur), then homologous sides cut in 3 aligned points, that is $ab \cap AB, bc \cap BC, ca \cap CA$ are collinear. The converse is true as well.

(Pascal theorem) If $p_1, p_2, p_3, q_1, q_2, q_3$ lie on a common conic in $\mathbb{P}$, then the three intersection points $p_1q_2 \cap p_2q_1$, $p_1q_3 \cap p_3q_1$ and $p_2q_3 \cap p_3q_2$ are aligned.
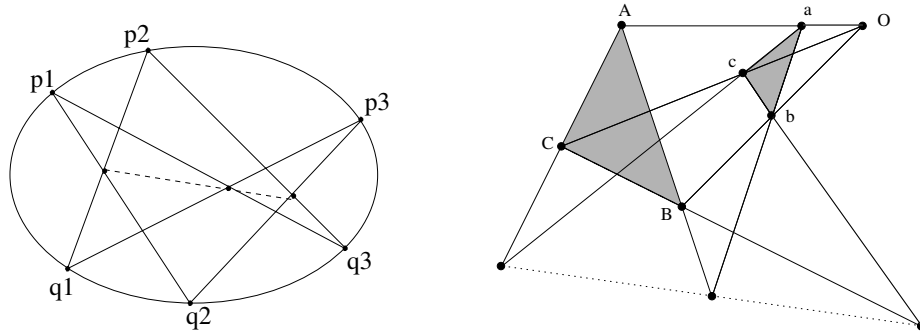

Fig. 3.   Pascal and Desargues theorems.

Note that Pascal theorem seems to get out our point/line framework, but recall that the union of two lines *is* a conic. Thus, Pappus theorem is a direct consequence of Pascal theorem. Moreover, we will see below that Pascal theorem can be re-formulated into the so-called hexamys theorem which falls into our framework.

All these properties are well-known theorems in the *real* projective plane, but, in fact, some of them are more fundamental than the others. For instance, it is a classical fact that Desargues property is a consequence of Pappus property. Our aim is to use these fundamentals properties in a combinatorial way to prove projective theorems.

Next section recalls the axiomatic framework of projective geometry.

2.3. *Axioms of projective geometry*

The real projective plane is not the only geometrical projective structure. In fact, geometers have axiomatized the projective framework giving the amazingly simple definition which follows.

Definition 1 (Projective geometry). A projective geometry is a set S of points and a collection of subsets of S, the set of *lines*, subject to these axioms:

   (P1) each pair A, B of distinct points is contained in a unique line which is denoted (AB),
   (P2) if A, B, C and D are distinct points for which $(AB) \cap (CD) \neq \emptyset$, then $(AC) \cap (BD) \neq \emptyset$, and
   (P3) each line contains at least three points.

For a projective *plane*, (P2) becomes: (P2') any two distinct lines intersect in exactly one point.

When three distinct points A, B and C belong to the same line, we say that they are collinear and we use the notation $[A, B, C]$.

Note that this definition has a matroidal formulation (see below). It comes directly from the definition that $[A, B, C]$, $[A, B, D]$ and $C \neq D$ implies that $[B, C, D]$. This fact is true in every projective geometry and it constitutes the simplest example of induced incidence. Note that Pappus and Desargues theorems are no longer true in this general framework, so we usually call them Pappus and Desargues *properties*.

From this definition, the following notions are usually derived:

- a *flat* is a set of points such that the entire line defined by two points A and B lies in the flat whenever A and B belong to it;
- the *flat closure* of a set E is the intersection of all flats containing E. It is the smaller flat containing E;
- the *rank* of a flat F is the minimal number of points of F necessary to cover F by flat closure. The respective ranks of a single point, a line and a plane are 1, 2 and 3.

Real $\mathbb{P}(\mathbb{R})$ or complex $\mathbb{P}(\mathbb{C})$ geometric projective planes are usual examples of projective planes. But not all projective planes arise from a field. We have the following characteristic properties:

Theorem 1 (Fundamental theorem of projective geometry[1,5,6]). *A projective plane is isomorphic to a projective plane arising from a division ring if and only if it satisfies Desargues property.*

A projective geometry is isomorphic to a projective geometry arising from a commutative field if and only if it satisfies the Pappus property.

Any projective geometry of a dimension greater than 2 is isomorphic to a projective geometry arising from a division ring (and Desargues property is true).

The main idea of the proof consists in defining addition and multiplication in terms of geometric constructions[1]. Numbers are represented by points along a particular line, where three distinct points are arbitrarily chosen, and called $0, 1$ and ∞. A geometric construction in Fig. 4 gives the point representing $a + b$, from the points representing a and b. For pedagogic reasons, the construction is first performed in the affine plane, using parallelism, then in the projective plane. The proof of the correction of this construction — for instance , the fact that the resulting point $a + b$ is independent of the used auxiliary points or lines, that the point $a + b$ equals the point $b + a$, that the addition so defined is indeed associative, etc. — uses Desargues property. Similarly, another construction in Fig. 5 gives the point representing ab, from the points representing a and b. Once again, the construction is first illustrated in the affine plane, then in the projective plane. The proof of the correction of this construction uses Desargues property again. The proof that the defined multiplication is commutative (that is permuting a and b gives the same result, $ab = ba$) uses the Pappus property.

6   *D. Michelucci and P. Schreck*
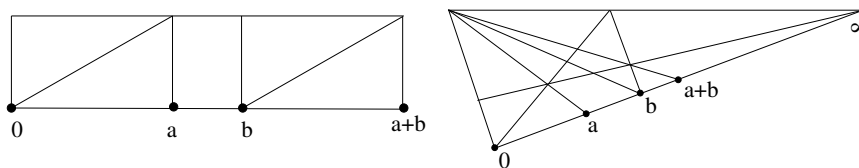
Fig. 4. the operation $a + b$. Affine and projective variants.
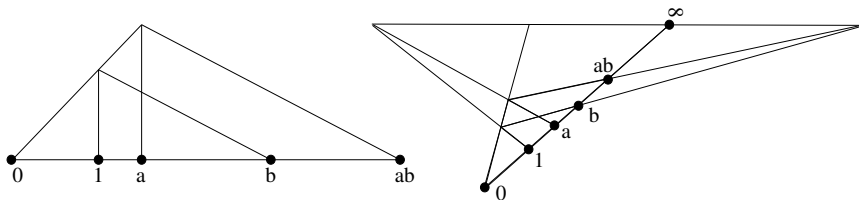
Fig. 5. The operation $a \times b$. Affine and projective variants.

This demonstration directly leads to the following theorem which states that incidence constraints are "universal".

Theorem 2 (Universality theorem). *All algebraic systems of equations with integer coefficients and unknowns in a field $\mathbb{K}$ (typically $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) reduce to a system of point-line incidence constraints in the projective plane $\mathbb{P}(\mathbb{K})$.*

Every integer coefficient is represented by a point on the line $0, 1, \infty$, and geometrically constructed from 0 and 1 with iterated squaring and multiplications. Every unknown of the algebraic system is also represented by a point on the line $0, 1, \infty$. Iterated squaring is also used to construct the point x^k for a given $k \in \mathbb{N}$. The point representing a monomial $x_1^{d_1} x_2^{d_2} \ldots$ is deduced by geometric multiplication of the points representing $x_1^{d_1}$, $x_2^{d_2}$, etc. The point representing a polynomial is the geometric sum of the points representing its monomials. Each equation $E_i = 0$ is represented by the equality between the points representing E_i and 0. Fig. 6 shows (a solution of) the point line incidence system of the equation $x^2 - 2 = 0$. The size of the resulting point line incidence system (the number of incidences) is proportional to the bit size of the algebraic system.

3. The compatible matroid problem

3.1. *Definition*

The matroid theory[19,16] intents to capture combinatorial properties of independence, which occur in graph theory, linear algebra, geometric arrangements. Several axiomatizations, cryptically isomorphic, were designed for matroids. The following axiomatization is convenient:
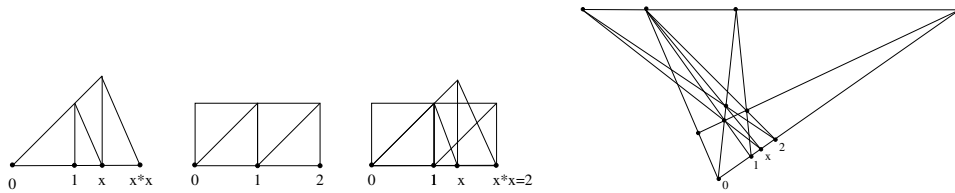
Fig. 6. From left to right: affine construction of the point x^2; middle: affine construction of the point 2; superposition of the two affine constructions; projective superposition, that is projective representation of the equation $x^2 - 2 = 0$.

- A set is either independent or dependent.
- The empty set is independent.
- Subsets of an independent set are independent.
- If the sets U and V are both independent, and V has one element more than U, then it is possible to complete U with one element $v \in V - U$ such that $U \cup \{v\}$ is independent.

A basic theorem of the matroid theory is that all maximal independent sets have the same cardinality, called the rank of the matroid. Matroids representing configurations in a projective plane (respectively in 3D) have rank 3 (resp. 4).

The compatible matroid problem can now be defined: given

- a set S of points,
- an integer K,
- a set I of subsets of S which have to be independent, and
- a set D of subsets of S which are specified to be dependent,

find a matroid with rank K and compatible with the given specification. The searched matroid can be represented by $O(|S|^R)$ boolean variables $\alpha(T)$, each one attached to a subset T of S with at most R elements with the meaning $\alpha(T)$ is true if and only if T is independent. The problem becomes "find values for the $O(|S|^R)$ boolean unknowns αs, compatible with the specification and the $O(|S|^{2R-1})$ axioms of matroids". Clearly, it relies on problem SAT, thus it is decidable and NP-complete.

This problem is relevant, since the non existence of a compatible matroid implies the configuration is not realizable. The converse is wrong: some matroids are not realizable in some (or all) fields.

3.2. *The compatible matroid problem is NP-complete*

The compatible matroid problem is NP-complete: the proof polynomially reduces the satisfaction problem[4] to a compatible matroid problem with rank 3. The satisfaction problem consists in finding values of boolean unknowns such that they satisfy a logical formula in conjunctive normal form as, for instance,
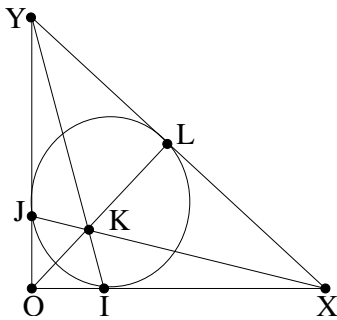
Fig. 7. The Fano plane, F_7: it is the projective plane with coordinates in $\mathbb{Z}/2\mathbb{Z}$.

$(x \vee y \vee \neg z) \wedge (\neg x \vee \neg y \vee z)$ (recall that $\vee$ is the disjunctive operator, $\wedge$ is the conjunctive operator, $\neg$ is the negation operator).

The satisfaction problem reduces to the compatible matroid problem in two steps. The first step classically consists in representing booleans by integers 0 and 1 in $\mathbb{Z}/2\mathbb{Z}$, $\wedge$ by the multiplication of $\mathbb{Z}/2\mathbb{Z}$, $x \vee y$ by $(x + (1 - x) \times y)$, and $\neg x$ by $1 - x$. Thus the problem is now to solve a set of algebraic non linear equations in $\mathbb{Z}/2\mathbb{Z}$.

In the second step, algebraic equations are translated into specifications of dependence or independence in the finite projective plane $\mathbb{P}(\mathbb{Z}/2\mathbb{Z})$ using the geometric constructions for $+$ and $\times$. $\mathbb{P}(\mathbb{Z}/2\mathbb{Z})$ is represented by (actually it is isomorphic to) the Fano matroid with vertices $\{O, I, X, J, Y, K, L\}$, and with minimal dependent subsets: $\{O, I, X\}$, $\{O, J, Y\}$, $\{O, K, L\}$, $\{X, Y, L\}$, $\{I, J, L\}$, $\{I, K, Y\}$, $\{J, K, X\}$, in Fig. 7. To help intuition, we give homogeneous coordinates for vertices: $O = (0, 0, 1), I = (1, 0, 1), X = (1, 0, 1), J = (0, 1, 1), Y = (0, 1, 0), K = (1, 1, 1), L = (1, 1, 0)$; O represent false, I represents true. Each boolean unknown is represented by a point. Each equation is represented by specifications. After solving the compatible matroid problem, each unknown boolean couple $(x_i, \neg x_i)$ is equal either to (O, I) (that is the set $\{x_i, O\}$ is dependent, the set $\{\neg x_i, I\}$ is dependent) or to (I, O), if a compatible matroid exists. If there is no compatible matroid, the formula is not satisfiable.

Thus, for ranks greater or equal to 3, the compatible matroid problem is NP-complete; for rank 2, it is only a reformulation of the union-find[4] problem, which has polynomial time complexity.

3.3. *Discussions about NP-completeness of compatible matroid problem*

Recall that the characterization of the rigidity of bar systems, which consist only of points and *generic* distance constraints, is the subject of the rigidity theory[15]. Incidence constraints and bar systems are two basic classes of geometric constraints. Both problems seem homologous but their difficulties strongly contrast. In 2D, sev-

eral deterministic methods allow to establish rigidity of bar systems in polynomial time, while the compatible matroid problem is NP-complete. In any dimension, a probabilistic method with cubic time compexity, decides the rigidity of bar systems.

Methods to decompose systems of geometric constraints in CAD-CAM are mainly graph based[2]. They have polynomial time complexity, and are very useful by enabling to solve systems which can not be otherwise. They are more or less inspired by methods from rigidity theory (for instance see Hendrickson's work[10]). A direct consequence of this NP-completeness is that these methods can give wrong results in presence of incidence constraints which are, unfortunately, ubiquitous in CAD-CAM applications.

Finally, due to its genericity assumptions, the rigidity theory is of little help to study incidence constraints. Admittedly, the witness configuration method[11] is directly inspired by the probabilistic method to test rigidity: interrogating the witness configuration makes possible to detect redundancies and decompose in polynomial time. However, this method assumes that a witness, that is a geometric realization of an incidence specification, is known. Thus the specification is obviously realizable, consistent, and the only possible dependences are redundancies, not contradictions.

4. The Hexamys prover

The Hexamys prover directly implements the proving method proposed by Raymond Pouzergues[12] and based on the following theorem.

Theorem 3 (Pouzergues hexamys theorem). *A hexamys is a hexagon such that its opposite sides cut in three collinear points. Every permutation of a hexamys is a hexamys.*

This theorem allows to discover new collinearity relations in a given figure, or to prove that certain collinearities are consequences of other collinearities (see two examples below). R. Pouzergues proves more than thirty theorems of geometry using a sequence of hexamys, incidence theorems but also theorems about conics. Using a clever projective reformulation of orthogonality, Pouzergues even proves Euclidean (that is metric) theorems involving orthogonality, like: the three altitudes of a triangle concur. Most of the time, a unique hexamys is sufficient for the proof; thus the proof is just 6 names long. But in more difficult cases, three hexamys are needed, as for the proof of Desargues in the Cevian case.

4.1. *Two examples*

Example 1. Fig. 8 depicts the Desargues theorem. Hexamys proves automatically this theorem in the generic case as follows. By hypothesis, OAA', OBB', OCC' are aligned. We want to prove the collinearity of IJK, that is [I,J,K] using our notation, where $I = AC \cap A'C'$, $J = AB \cap A'B'$, $K = BC \cap B'C'$. The auxiliary points: $P = A'B' \cap BC$, $Q = AB \cap B'C'$ are needed. Now (A, A', P, C, C', Q) is a hexamys since its three opposite sides cut in O, B, B', collinear by hypothesis. Thus
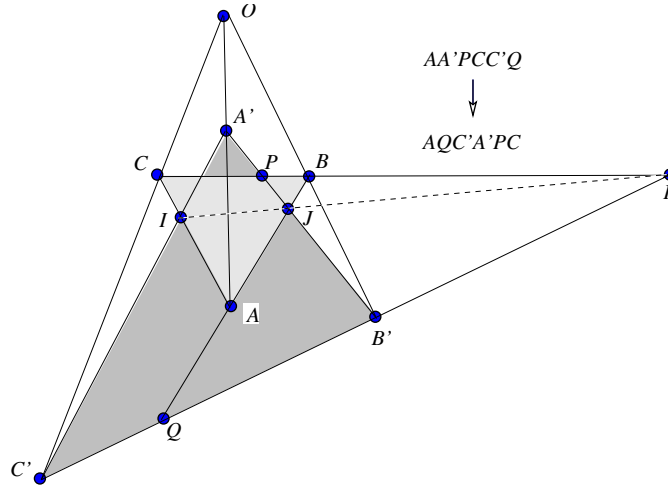
10   *D. Michelucci and P. Schreck*

Fig. 8. Desargues property is a consequence of the hexamys property. (This proof of the generic case has been automatically generated).

(A, Q, C', A', P, C) is a hexamys as well, the opposite sides cut in J, K, I, thus they are collinear. QED.

Example 2. Desargues theorem is true even in the "Cevian case" depicted on Fig. 9. Recall that $A'B'C'$ is a Cevian triangle of ABC if $A' \in (BC)$, $B' \in (AC)$, $C' \in (AB)$ and (AA'), (BB') and (CC') are concurrent. We interactively prove the theorem as follows.

We have to prove that P, Q, R are collinear where $P = (AB) \cap (A'B')$, $Q = (AC) \cap (A'C')$ and $R = (BC) \cap (B'C')$. We introduce $R_1 = (PQ) \cap (BC)$ and $R_2 = (PQ) \cap (B'C')$: we have now to prove that $R_1 = R_2$. Auxiliary intersection points are $I = (OB) \cap (PQ)$, $L = (A'C') \cap (AI)$, $N = (AB) \cap (LB')$ and $M = (AL) \cap (CC')$ (the existence of these points is a question left to the reader). Since $[L, N, B']$, (A', L, B', I, A, P) is a hexamys, so is (A', A, P, I, B', L), and we have $[O, N, Q]$ by using only one hexamys. With the second hexamys, (C, B, A, O, I, Q) which gives by permutation (C, O, Q, I, A, B), we prove that $[R_1, M, N]$ A third hexamys is used to prove $[R_2, M, N]$: this is (C', B', Q, O, I, L) giving (C', O, Q, I, L, B'). Thus the lines (MN) and (PQ) have two points in common: R_1 and R_2. Either the lines are equal but then the configuration degenerates and all points are aligned; or $R_1 = R_2$ and P, Q and $R = R_1 = R_2$ are collinear. Three hexamys and three "generations" of lines are needed here.

4.2. *Hexamys rules*

The following genericity assumption of the Hexamys prover simplifies dramatically the problem, and the prover itself: two points with different names are independent (that is distinct). The genericity assumption enables to use the generality of the
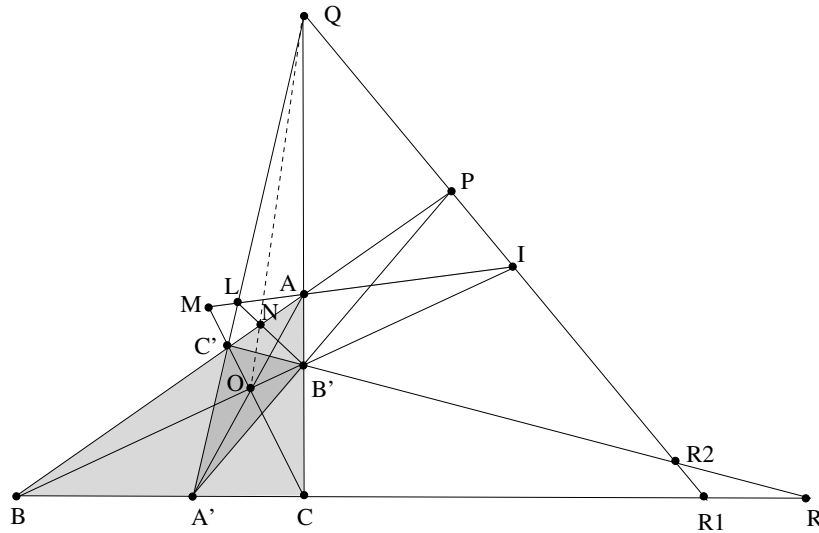
Fig. 9. Cevian triangles

hexamys theorem, which has only one non degeneracy condition: the six points (a, b, c, d, e, f) must be distinct; indeed, the hypothesis of the hexamys theorem do not require independences for triples of points in (a, b, c, d, e, f).

The user specifies a set of collinearities between three distinct points, that is between three distinct names of points. It is not possible (and useless) to specify that three points are not collinear. The Hexamys prover uses the following rules:

- a matroid rule: $[a, b, c] \wedge [a, b, d] \wedge c \neq d \Rightarrow [b, c, d]$.
- the hexamys permutation theorem, with σ any permutation:

$$\text{Hex}(a, b, c, d, e, f, g) \Leftrightarrow \text{Hex}(\sigma(a, b, c, d, e, f, g))$$

- a convenient definition: $p = a \ \vee \ p = b \ \vee [p, a, b] \Leftrightarrow p \in (a, b)$
- the hexamys theorem provides two rules:

$$\left.\begin{array}{l} a, b, c, d, e, f \text{ all different} \\ \wedge \ i \in (a, b) \ \wedge \ i \in (d, e) \\ \wedge \ j \in (b, c) \ \wedge \ j \in (e, f) \\ \wedge \ k \in (c, d) \ \wedge k \ \in (f, a) \end{array}\right\} \Rightarrow \text{Hex}(a, b, c, d, e, f, g)$$

$$\left.\begin{array}{l} i, j, k \text{ all different} \\ \wedge \ \text{Hex}(a, b, c, d, e, f, g) \\ \wedge \ i \in (a, b) \ \wedge \ i \in (d, e) \\ \wedge \ j \in (b, c) \ \wedge \ j \in (e, f) \\ \wedge \ k \in (c, d) \ \wedge k \ \in (f, a) \end{array}\right\} \Rightarrow [i, j, k]$$

This last rule uses property $i \in (a, b)$ instead of $[i, a, b]$: this is because the latter implies that i, a, b are all distinct, contrarily to $i \in (a, b)$. The rules above can not create collinearities $[a, b, c]$ where a, b, c are not all distinct (names of) points, starting from a specification where all collinearities involve triples of distinct points.

¿From these rules and the initial specification of collinearities, it is easy to deduce new hexamys and new collinearities, using either forward chaining (we implemented a saturation method using Ocaml language) or backward chaining (we implemented it in Prolog language). We reproduce Pouzergues proofs and some others.

4.3. *Hexamys features*

For genericity reasons, the Hexamys prover needs no backtrack. For instance, it can not happen that the prover "believes" or supposes that a triple (a, b, c) is independent, deduces new hexamys and collinearities, and then realizes that the independence of (a, b, c) is actually wrong. It is due to the generality of the Hexamys theorem and of the genericity assumption.

For the same reasons, the Hexamys can never find that a specification is inconsistent: when two lines cut in two points, the Hexamys prover deduces the two lines are equal, so that, in the extremal case, all points become collinear. It is worth to point out this asymmetry between lines and points. The behavior of Hexamys gives a strategy to the user for proving that two points are equal: create two distinct points, then check that Hexamys merges all lines through them.

Always for these reasons, the Hexamys prover is polynomial time: $O(n^6)$ for n initial and auxiliary points. Of course, this prevents the Hexamys prover to treat problems arising from the geometrization of satisfaction problems, as in the NP-completeness proof of the compatible matroid problem, where the genericity assumption is wrong.

¿From an axiomatic viewpoint, all theorems in the projective plane results from the Pappus axiom and the "matroidal" axioms. But, in fact, the Hexamys prover has it own limitations.

The Hexamys prover is only half deterministic since it can only prove collinearities of points but not non-collinearities. Note that the later notion does not even exist in the Hexamys world, though it is relevant for the user. Moreover, adding auxiliary intersection points is essential to enable Hexamys to prove collinearities. If only one hexamys is needed, the auxiliary points are automatically found and created. If more hexamys are needed, the user has to interact with the prover in order to guide it. Of course, it is possible to systematically generate intersection points; but it has an exponential cost and we do not know stopping criteria. To overcome these limitations, we come back to compatibles matroids and their formulation in term of SAT problem.

5. Pappus prover

5.1. *The compatible configuration problem*

Some rank 3 matroids are not realizable in projective fields, for instance they contain configurations which contradict theorems such as Desargues, Pappus, or hexamys. This suggests to search matroids which do not contain forbidden configuration (forbidden minors[16,19], in matroid parlance). We call this problem, the compatible configuration problem. The aim of our Pappus prover is to check such configurations by searching a rank 3 matroid

- which is compatible with a specification given by the user under the form of a set of dependent tuples and a set of independent tuples,
- and which is not contradictory with Pappus property.

5.2. *Rules of Pappus prover*

As said in section 3.1, we can define a boolean variable $\alpha(T)$ for each subset T with three points or less by using the intuitive semantic: $\alpha(T)$ is true if and only if T is independent. For n points, there are $O(n^3)$ boolean variables. Every subset S with more than three points is dependent, so variables $\alpha(S)$ are useless for $|S| > 3$. The axioms of the theory of matroids and Desargues and Pappus axioms provide a boolean formula in conjunctive normal form (CNF) which is solved with a boolean solver, namely Minisat[8]. Minisat is fast enough to solve configurations with 25 points in a matter of seconds, but it crashes for some configurations involving 34 points or more.

For every subset B, $|B| \leq 3$, and $A \subset B$: $\alpha(B) \Rightarrow \alpha(A)$. The corresponding clause is: $\neg\alpha(B) \vee \alpha(A)$. For every couple of subsets U, V, where $|V| \leq 3$ and $|V| = |U| + 1$, the complementing axiom of matroids asserts that, if U and V are independent, then it is possible to complete U with an element in $V - U$ to get an independent set: $\alpha(U) \wedge \alpha(V) \Rightarrow \alpha(U + v_1) \vee \alpha(U + v_2) \ldots \vee \alpha(U + v_k)$ where $v_i \in V - U$ and $U + v_i$ is a shortcut for $U \cup \{v_i\}$. These axioms are translated into clauses as usual. For rank 3, the matroid theory provides $O(n^5)$ clauses for n points.

Forbidden configurations contradicting Pappus' or Desargues' theorems provide another set of clauses. These clauses involve every subset of 10 points, thus they are terribly numerous; the major part of them is useless. To speed up, only the matroidal part is solved in a first stage; if Minisat finds a solution, the Pappus and Desargues clauses violated in the solution are added to the formula ($O(n^6)$ operations are needed to find them), and the boolean solver Minisat is called again. This loop is repeated until no more Pappus or Desargues clause is violated, or the boolean formula becomes unsatisfiable.

It is worth mentioning that the specification is the only non symmetric part of the boolean formula: all matroid, Pappus and Desargues clauses are permutation invariant. Minisat does not exploit this high level of symmetry.

A theorem: $H \Rightarrow C$ is proved by refutation, that is if and only if $H \wedge \neg C$ is contradictory. We check Pappus on a subset of the theorems proved with Hexamys.

5.3. *Features of Pappus prover*

The Pappus prover is typically faster than Hexamys but we will not discuss further running times, since these provers are only prototypes. Pappus provides strictly no explanation, while Hexamys gives short and readable proofs. Moreover, the specifications used by Pappus prover have to be complete, that is all non degeneracy conditions must be specified –which is error prone, and a burden for the user.

As the Hexamys prover, the Pappus prover is only half deterministic. More precisely, if a specification is not compatible, adding points and specifications will not make it compatible. Conversely, if no contradiction is detected, it does not guarantee that the configuration is compatible "for ever": maybe a forbidden configuration is missing – or some non degeneracy condition has been forgotten in the specification. As for Hexamys prover, adding an auxiliary intersection point can enable the Pappus prover to detect a contradiction which was not detected before. Each time the Pappus prover does not detect a contradiction it should detect, it means that a new forbidden configuration is discovered. Identifying minimal forbidden configurations is a work in progress.

¿From matroid theory[19,16], there is an infinite quantity of minimal forbidden configurations for realizability in the reals. However only a finite number is relevant. So, we only need to study configurations with a finite number of points. The possibility of detecting or automatically learning all relevant forbidden configurations is an open question. Clearly, if we can detect all forbidden configurations, the Pappus prover can be made fully deterministic.

It would give an exotic method to solve the basic problem of computer algebra, the ideal membership problem: given polynomials g_i and a polynomial f all in $\mathbb{Q}[X]$, determine if there are polynomials $h_i \in \mathbb{Q}[X]$ such that $f = \sum h_i \times g_i$, in other words if $g_i(X) = 0 \Rightarrow f(X) = 0$. Just translate the algebraic question into a question about incidence constraints and use the Pappus prover. The NP-completeness of the proving problem is not an argument to reject such an exotic approach since the complexity is intrinsic to the problem.

6. Projective theorems and proofs in higher dimensions

This section conjectures that it is possible to reach full determinism in combinatorial provers.

6.1. *Non monotony*

A first common limitation to Pappus and Hexamys provers consists in their restriction to the plane. Indeed, no extension of Pascal or hexamys theorems is known concerning the 3D space and beyond.

Another remaining limitation is that both provers are only half deterministic up to now. According to Hilbert, or to the universality theorem, all theorems of the projective plane are consequences of Pappus axiom and of the simple axioms for rank 3 matroids. This gap between logic and provers comes from the non matroid axioms: Desargues, Pappus or hexamys, used in forbidden configurations. Indeed, these axioms introduce non monotony in the following sense: completing a compatible configuration with a missing intersection point can make the configuration non compatible. On the other hand, we conjecture that the matroid theory is monotone, more precisely, that it is always possible to complete a matroid with a missing intersection point without introducing a contradiction. Now, Pappus theorem is captured by matroids over the cubic curves in the projective plane (see proof below). Suppose that the user specifies incidences between points and cubic curves in the projective plane [a]. The idea is to search a rank 10 matroid compatible with the specification. Such a matroid will automatically satisfy Pappus axiom, and, due to the conjectured monotony of matroids, adding missing intersection points will not make the compatible matroid inconsistent.

Even if this conjecture is correct, the method is clearly not practicable: for n points, there are $O(n^{10}/10!)$ tuples of at most 10 points, thus a satisfaction problem involving $O(n^{10}/10!)$ boolean unknowns has to be solved. This method is only an interesting but theoretical argument.

6.2. *Chasles theorem and its consequences*

Assuming that $\mathbb{P}$ is a projective plane arising from a field, we have:

Theorem 4 (Chasles). *If Q_1 and Q_2 are two cubic curves in $\mathbb{P}$ without a common component, which cut in 9 distinct points, then every cubic passing through 8 of these points also passes through the 9th point.*

Numerous theorems[5,6,7,18] share the same combinatorial flavor like, for instance, the three quadrics theorem, Beltrami theorem and Mobius's tetrahedrons theorem. A matroidal proof of Chasles theorem is given next section. We prove before that Pappus property is a consequence of Chasles theorem:

Let Q_1 be the cubic curve composed of the 3 lines p_1q_2, p_2q_3, p_3q_1. Let Q_2 be the cubic curve composed of the 3 lines p_2q_1, p_3q_2, p_1q_3. Let C be the conic composed of the 2 lines $p_1p_2p_3$ and $q_1q_2q_3$. Let K be the cubic $C \cup i_3i_1$ where $i_3 = p_1q_2 \cap p_2q_1$, $i_1 = p_2q_3 \cap p_3q_2$. We note $i_2 = p_3q_1 \cap p_1q_3$. The two cubic curves Q_1 and Q_2 cut in the 9 points $p_k, q_k, i_k, k \in 1, 2, 3$. The cubic K passes through all these points too, except i_2. By Chasles theorem, since it passes by 8 of the 9 points, it also passes by the 9th point i_2. Thus $K = p_1p_2p_3 \cup q_1q_2q_3 \cup i_3i_1$ passes through i_2. But i_2 does not lie on $p_1p_2p_3 \cup q_1q_2q_3$, thus it lies on i_3i_1, and i_1, i_2, i_3 are aligned. QED.

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[a]to give an idea, the table 10 lists the quadratic rank of basic configurations of points. The table for cubic ranks is defined similarly but is much bigger.

16   *D. Michelucci and P. Schreck*

R_3	R_6	coR$_6$	Configuration
$R_3 = 1$	$R_6 = 1$	coR$_6 = 5$	1 point
$R_3 = 2$	$R_6 = 2$	coR$_6 = 4$	2 points
$R_3 = 3$	$R_6 = 3$	coR$_6 = 3$	3 non aligned points
$R_3 = 2$	$R_6 = 3$	coR$_6 = 3$	3 aligned points
$R_3 = 3$	$R_6 = 4$	coR$_6 = 2$	4 points in generic position
$R_3 = 3$	$R_6 = 4$	coR$_6 = 2$	4 points, 3 aligned
$R_3 = 2$	$R_6 = 3$	coR$_6 = 3$	4 aligned points
$R_3 = 3$	$R_6 = 5$	coR$_6 = 1$	5 points, in generic position
$R_3 = 2$	$R_6 = 4$	coR$_6 = 2$	5 aligned points
$R_3 = 3$	$R_6 = 4$	coR$_6 = 2$	5 points, 4 aligned
$R_3 = 3$	$R_6 = 5$	coR$_6 = 1$	5 points, 3 aligned
$R_3 = 3$	$R_6 = 6$	coR$_6 = 0$	6 points, in generic position
$R_3 = 3$	$R_6 = 5$	coR$_6 = 1$	6 points, on a conic, generically
$R_3 = 3$	$R_6 = 5$	coR$_6 = 1$	6 points, 3 aligned + 3 aligned
$R_3 = 3$	$R_6 = 5$	coR$_6 = 1$	6 points, 4 aligned
$R_3 = 3$	$R_6 = 5$	coR$_6 = 1$	6 points, 5 aligned
$R_3 = 2$	$R_6 = 4$	coR$_6 = 2$	6 aligned points

Fig. 10. R_3 is the linear rank, R_6 the conic rank, coR$_6$ the conic corank. The latter is the number of independents conics through the points. $R_6 + \text{coR}_6 = 6$. In the same way, it is possible to tabulate the cubic ranks.

The proof of Pascal theorem is quite similar: just replace the conic C by the conic passing through the 6 points $p_k, q_k, k \in 1, 2, 3$.

6.3. *Matroidal proof of Chasles theorem*

Some notations are needed in order to present succinctly the proof of Chasles theorem and the ideas behind the proof. Points in the projective plane are represented by 3-vectors $v = (x, y, z)$. As usual, all proportional non vanishing vectors represent the same point in the projective plane. Homogeneous coordinates x, y, z will be used only for counting degrees of freedom. We define a cubic lifting ϕ as a map transforming 3-vector $v = (x, y, z)$ into 10-vector

$$\phi(x, y, z) = (x^3, y^3, z^3, xyz, x^2y, x^2z, y^2z, xy^2, xz^2, yz^2)$$

Let Q be a cubic curve in the projective plane, $\phi(Q)$ is then an hyperplane in a 10 dimensional vectorial space. Thus, $\phi(Q)$ has vectorial rank nine, and nine points are needed to define a cubic curve. If the nine points are in generic position, they define one and only one cubic curve.

We can now outline the proof of Chasles theorem. By hypothesis, Q_1 and Q_2 are two cubic curves in the projective plane which do not share common component, and which intersect in 9 distinct points. Therefore, $\phi(Q_1)$ is an hyperplane in the lift space of dimension 10, and $\phi(Q_1)$ has then vectorial rank equal to 9. This is the same for Q_2. The intersection of two hyperplanes with rank 9 has rank 8. Thus, any set of 9 vectors of the intersection of $\phi(Q_1)$ and $\phi(Q_2)$ is linearly dependent.

Consequently, every vectorial space, such as an hyperplane, through 8 of them also contains the remaining 9th. In other words, coming back to the plane, every cubic curve through 8 of the 9 intersection points goes through the remaining 9th. QED.

The previous proof uses only properties of matroids: it uses no forbidden configurations, which introduce non monotony.

7. Conclusion

Detecting dependences between geometric constraints is difficult. Indeed the easiest problem which is the compatible matroid problem, is NP-complete. This complexity is not due to some flaw in the combinatorial approaches, which on the contrary provides the proof of this complexity. This complexity is intrinsic to the problem and holds for all methods. Some mild assumptions, such as genericity, make possible polynomial time methods, such as the Hexamys prover.

Incidentally, several open questions arise. Is it possible to make our half deterministic provers completely deterministic? Are forbidden configurations detectable? Does it make sense, *computationally*, to translate computer algebra problems (for instance the ideal membership problem) into problems of point-line incidences? Is the theory of matroid theory monotone?

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18   *D. Michelucci and P. Schreck*

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