Kernel functions and coordinate-free equations of implicit curves and surfaces
Work in Progress

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Abstract

Recently several authors propose to use coordinate free formulations of geometric constraints. Thus the question arises to formulate in such a way all geometric constraints, such as: six 2D points must lie on a common conic curve, ten 2D points must lie on a common 2D cubic, ten 3D points must lie on a common quadric, etc. This draft gives (for the first time, as far as we know) an intrinsic condition involving only scalar products of vectors, first, for coplanar points to lie on a common degree d algebraic curve, and, second, for 3D points to lie on a common degree d algebraic surface. Kernel functions are used; they permit to extend relations à la Cayley Menger.

1 Introduction

Several researchers independently proposed to use intrinsic (coordinate free) formulations of geometric constraints, for resolution: Lu Yang [Yan03, Yan02] generalizes Cayley Menger relations; Alain Rivière, André Clément, Philippe Serré, Auxkin Ortuzar, David Lesage, Jean-Claude Léon [Ser00, Les02, LLS00, CRS99, SCR02, SCR03, JBL+11, SOR06] use the metric tensor formalism; and us [MF04], in an approach similar to Lu Yang's one; in peculiar, we show that the Cayley Menger determinants bring an elegant solution to the Stewart platform (sometimes called the octahedron problem, *eg* in [HY01]). See also [MS12] for a recent article in this wake.

Instead of using cartesian coordinates, or other kinds of coordinates like Grassmann-Plucker coordinates or some Clifford algebra, intrinsic formulations of geometric constraints involve only parameters which are independent of systems of coordinates, namely distances or angles, scalar products, (signed) areas or volumes, cross ratios. This latter approach is also used in some provers, typically the *area method* by Chou, Gao and Zhang[CGZ94, CGZ93] or Havel's approach [Hav91]. For conciseness, the reader is refered to the previous papers, some of which detail the advantages of intrinsic formulations.

This intrinsic approach poses the problem of expressing all geometric constraints in a coordinate free way, which is not always easy; indeed, the cartesian formulation is much more widespread and benefits from more than two centuries of mathematical study.

This draft shows how the kernel functions formalism, recently used by Computer Scientists for SVM (Support Vector Machines: they are classifiers, introduced by Vapnik and Chervonenkis [CST00]), can help to provide intrinsic formulations. The latters will involve only distances between points and scalar products between vectors, in the wake of the metric tensor (Gram matrix) approach already mentionned [Ser00, Les02, LLS00, CRS99, SCR02, SCR03]. Thus they are independent of all coordinate systems.

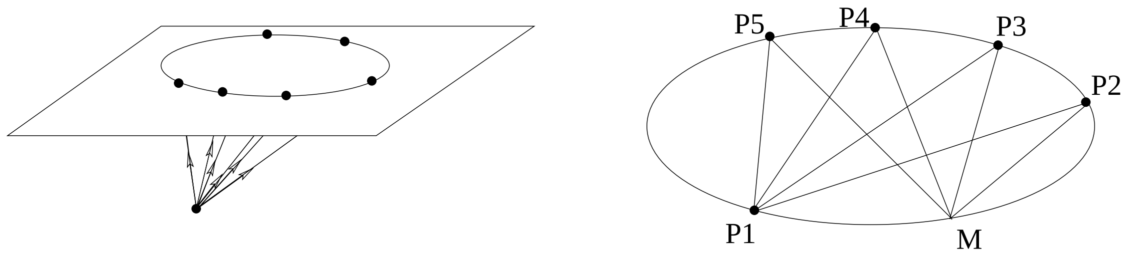

Figure 1: Vectorial condition for points to lie on a common conic or algebraic curve with degree d? Left: first case; right: second case.

2 First case

Look at Fig. 1: what is the condition on the scalar products between vectors v_i (they have any norm) for the corresponding intersection points between the lines they support and some plane (any plane *not passing through the vectors origin*) to lie on the same conic? This paper shows that the matrix M with $M_{i,j} = (v_i \cdot v_j)^2 = M_{j,i}$ must have rank 5, or less. If the $n \geq 6$ intersection points do not lie on the same conic, but are generic, the matrix M has rank 6 (assuming the v_i lie in 3D space). More generally:

Th1. The intersection points between some plane π and the lines defined by supporting vectors v_i through a common origin Ω outside π lie on a degree d curve iff the matrix $M^{(d)}$, where $M^{(d)}_{i,j} = (v_i \cdot v_j)^d = M^{(d)}_{j,i}$, has rank $r_d = d(d+3)/2$ or less (r_d for deficient rank). The generic rank $g_d = r_d + 1 = (d+1)(d+2)/2$ (the rank of the matrix in the generic case) is given by the number of monomials in the polynomial in 2 variables of degree d, since this curve is the zero set of such a polynomial. The proof uses kernel functions.

3 Kernel functions

First an example. Consider six points in 2D with homogeneous coordinates $p_i = (x_i, y_i, h_i)$. They are lifted to $P_i = \phi(p_i) = (x_i^2, y_i^2, h_i^2, x_i y_i, x_i h_i, y_i h_i)$. By definition of conics, if the p_i lie on a comon conic $ax_i^2 + by_i^2 + ch_i^2 + dx_i y_i + ex_i h_i + fy_i h_i = 0$, then the P_i lie on a common hyperplane, having equation: $P_i \cdot h = 0$ with $h = (a, b, c, d, e, f)$. Thus six generic (in other words, random, and thus not lying on a common conic) 2D points p_i give six lifted points P_i with rank 6. Six 2D points p_i lying on a common conic give six lifted points P_i with rank 5 (or even less, but this degenerate case is not considered for short).

So the arising idea is to lift points p_i to $\phi(p_i) = P_i$ in higher dimensional space, to compare the rank of the P_i with the generic rank, which is by definition the rank of generic $\phi(g_i)$, where g_i lie in the same space than the p_i but are random.

We need two other notions: Gram matrices and kernel functions.

First notion: If m vectors $P_1, \ldots P_m$ have rank r, their Gram matrix: $G_{ij} = P_i \cdot P_j = G_{ji}$ has also rank r. In passing, if a square matrix (such as a Gram matrix) is rank deficient, classical linear algebra permits to compute explicitly the linear dependence relations between its rows.

Second notion: to compute $P_i \cdot P_j$, the naive method compute $P_i = \phi(p_i)$, and $P_j = \phi(p_j)$, then $P_i \cdot P_j$. Kernel functions permit to avoid the computation of $\phi(p_i)$. A kernel function K is such that $K(p_i, p_j) = \phi(p_i) \cdot \phi(p_j)$. Here are two examples.

Example 1. $p = (x, y, h)$ et $\phi(p) = (x^2, y^2, h^2, \sqrt{2}xy, \sqrt{2}xh, \sqrt{2}yh)$. The $\sqrt{2}$ factors do not modify ranks, and permit to say:

$$\begin{aligned} K(p, p') &= \phi(p) \cdot \phi(p') \\ &= (x^2, y^2, h^2, \sqrt{2}xy, \sqrt{2}xh, \sqrt{2}yh) \cdot (x'^2, y'^2, h'^2, \sqrt{2}x'y', \sqrt{2}x'h', \sqrt{2}y'h') \\ &= x^2x'^2 + y^2y'^2 + h^2h'^2 + 2xyx'y' + 2xhx'h' + 2yhy'h' \end{aligned}$$

$$= \quad (xx' + yy' + hh')^2 = (p \cdot p')^2$$

More generally, for a homogeneous polynomial kernel of degree d, $K(p, p') = (p \cdot p')^d$. It is used in SVMs ([CST00] for a proof). Hint: it suffices to adjust the monomial coefficients, like $\sqrt{2}$ in the example. Thus the Gram matrix for this homogeneous lift with degree d is: $G_{i,j} = (p_i \cdot p_j)^d$.

The proof of theorem **Th1** follows straightforwardly.

In this following second example of kernel functions, we consider non homogeneous polynomial liftings, to complete this micro survey on polynomial kernel functions. Define $p = (x, y)$ and $\phi(p) = (x^2, y^2, \sqrt{2}xy, \sqrt{2}x, \sqrt{2}y, 1)$. The $\sqrt{2}$ does not modify rank, and permits to say that:

$$\begin{aligned} K(p, p') &= \phi(x, y) \cdot \phi(x', y') \\ &= (x^2, y^2, \sqrt{2}xy, \sqrt{2}x, \sqrt{2}y, 1) \cdot (x'^2, y'^2, \sqrt{2}x'y', \sqrt{2}x', \sqrt{2}y', 1) \\ &= x^2x'^2 + y^2y'^2 + 2xyx'y' + 2xx' + 2yy' + 1 \\ &= (xx' + yy' + 1)^2 = (p \cdot p' + 1)^2 \end{aligned}$$

More generally, for a polynomial non homogeneous lifting of degree d, $K_(p, p') = (p \cdot p' + 1)^d$. It is used in SVMs. Thus the Gram matrix for this lift with degree d: $G_{i,j} = (p_i \cdot p_j + 1)^d$. This formula will appear in the next section, when considering the "second case".

4 Second case

What is the coordinate-free condition for six 2D points $P_i, i = 0 \dots 5$ to lie on a common quadric, or on a common algebraic curve with degree d? This time, we search a condition involving scalar product between vectors P_0P_j, thus independent on the coordinates of the P_is.

Note: actually, for conics, several intrinsic and classical conditions are already available (see section 7) and can legitimately be used. However these latter formulations require geometric elements which may not be part of the initial problem (e.g. foci or directrix), and they do not seem to easily extend to higher degrees. Thus the following formulation can be of interest.

The P_i plane π is embedded in 3D space: let Ω be any one of the two points such that ΩP_0 is orthogonal to π, and the distance ΩP_0 equals 1. Then we use theorem **Th1**: the P_i lie on the same conic iff the matrix M, where $M_{i,j} = (\overrightarrow{\Omega P_i} \cdot \overrightarrow{\Omega P_j})^2$, has rank five or less, and the P_i lie on the same algebraic curve with degree d iff the matrix M, where $M_{i,j} = (\overrightarrow{\Omega P_i} \cdot \overrightarrow{\Omega P_j})^d$ has deficient rank $r_d = d(d+3)/2$. This formulation regrettably requires the point Ω, which is not part of the initial problem, but to remove it, it suffices to see that:

$$\begin{aligned} \overrightarrow{\Omega P_i} \cdot \overrightarrow{\Omega P_j} &= (\overrightarrow{\Omega P_0} + \overrightarrow{P_0 P_i}) \cdot (\overrightarrow{\Omega P_0} + \overrightarrow{P_0 P_j}) \\ &= \overrightarrow{\Omega P_0} \cdot \overrightarrow{\Omega P_0} + \overrightarrow{\Omega P_0} \cdot \overrightarrow{P_0 P_j} + \overrightarrow{P_0 P_i} \cdot \overrightarrow{\Omega P_0} + \overrightarrow{P_0 P_i} \cdot \overrightarrow{P_0 P_j} \\ &= 1 + 0 + 0 + \overrightarrow{P_0 P_i} \cdot \overrightarrow{P_0 P_j} \end{aligned}$$

Th2: the points P_i, coplanar, lie on the same algebraic curve with degree d iff the matrix M has deficient rank (*ie* $r_d = d(d+3)/2$) or less, where $M_{i,j} = (1 + \overrightarrow{P_0 P_i} \cdot \overrightarrow{P_0 P_j})^d$.

Note: **Th2** applies when $d = 1$. Three points P_0, P_1, P_2 are collinear iff

$$M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 + \overrightarrow{P_0 P_1} \cdot \overrightarrow{P_0 P_1} & 1 + \overrightarrow{P_0 P_1} \cdot \overrightarrow{P_0 P_2} \\ 1 & 1 + \overrightarrow{P_0 P_2} \cdot \overrightarrow{P_0 P_1} & 1 + \overrightarrow{P_0 P_2} \cdot \overrightarrow{P_0 P_2} \end{pmatrix}$$

has rank 2 (or less). Substracting the first row to the two other rows, then substracting the first column to the two other columns does not modify the rank, and gives the Gram matrix G of vectors $\overrightarrow{P_0 P_1}, \overrightarrow{P_0 P_2}$. We get that $|M|$ vanishes iff $|G|$ vanishes: correct. The equivalence between the Gram matrix $G(\overrightarrow{P_0 P_1}, \overrightarrow{P_0 P_2} \dots)$ and the Cayley Menger matrix $CM(P_0, P_1 \dots)$ (defined in [MF04]) is also well known. Thus **Th2** is consistent with the results obtained with Cayley Menger determinants in [MF04]. The same considerations hold for **Th1**, of course.

5 The circular case

Cayley Menger determinants [MF04] give a relation for four 2D points to be cocyclic (to lie on the same cercle), and for five 3D points to be cospheric (to lie on the same sphere). Considering only the 2D case for short, theorems **Th1** and **Th2** gives another intrinsic formulation for four points to lie on the same cercle, using the two "cyclic points". By definition, cyclic points lie on all cercles; they have homogeneous coordinates $I = (1, i, 0)$ and $J = (1, -i, 0)$ (where i is the imaginary number such that $i^2 = -1$) in all real cartesian systems of coordinates. These considerations extends to cosphericity in 3D. Of course, these intrinsic formulations entail temporary computations with complex numbers, which may be unconvenient.

6 Implicit formulation for surfaces

This theory extends nicely to 3D surfaces (and beyond). The points P_i, lying in 3D space, lie on the same algebraic surface of degree d iff the matrix M has deficient rank $r_d = g_d - 1$ (or less), where $M_{i,j} = (1 + \overrightarrow{P_0P_i} \cdot \overrightarrow{P_0P_j})^d$. The generic rank g_d is the rank of the matrix M for generic (*ie* random) 3D points; it is $\binom{3}{d+3} = \binom{d}{d+3}$ (or the number of points, if there is less than $\binom{3}{d+3}$ points).

7 Appendice: Intrinsic formulations for conics

This appendice tersely lists classical and well known intrinsic (coordinate free) definitions of conics. First, according to the Pascal theorem, or the hexamy theorem, 6 points lie on a common conic if the opposite sides of their hexagon (for any cyclic order of the 6 vertices) cut in 3 collinear points. Note in passing that the corresponding geometric characterization in 3D for 10 points to lie on the same quadric is still unknown.

Second, there is also the formulation based on cross ratios (which is maybe more easily generalizable?): a point M lie on the conic defined by the five points $P_1, \ldots P_5$ iff the cross ratio of the lines MP_2, MP_3, MP_4, MP_5 equals the cross ratio of the lines P_1P_2, P_1P_3, P_1P_4, P_1P_5 (Fig. 1, right). In passing, we meet here what is perhaps a limitation of the metric tensor approach: the latter considers that unknowns are scalar products, *ie* lengths and cosinus; since $\cos(u, v) = \cos(v, u)$, and $\sin(u, v) = -\sin(v, u)$ and since signed areas and cross ratios (which are ratios of signed areas) are determined by sinus, it turns out that the metric tensor approach prevents itself to use sinus, signed areas and cross ratios. Note that another intrinsic approach: the area method of Gao, Chou and Zhang makes an opposite choice. But the area method works only in 2D.

Third, there is the definition by directrix (some line) and focus (some point outside the directrix): the conic is the locus of points whose distance from the focus is proportional to the distance from the directrix: according to the ratio is less than/ equal to/ greater than 1, the conic is an ellipse/ a parabola/ an hyperbola. Other intrinsic definitions can be deduced for ellipses (loci of points M such that $F_1M + MF_2 = 2a$, where F_1, F_2 are the two foci, and $2a \geq F_1F_2$ some constant) and hyperbolas (loci of points M such that $|MF_1 - MF_2| = 2a$, F_1, F_2 are foci and $2a \leq F_1F_2$ some constant). Other intrinsic formulations are known, which typically consider homographies.

8 Conclusion

This short paper gives, for the first time, an intrinsic formulation, involving only scalar products, for coplanar points to lie on a common degree d algebraic curve, for 3D space to lie on a common degree d algebraic surface, etc. These conditions are very simple and symmetric, and use the theory of kernel functions. Of course, the corresponding condition, in the usual cartesian formulation, is known since more than one century: this shows that, indeed, the intrinsic approach is still in its infancy when compared to the cartesian formulation.

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