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Using Cayley-Menger Determinants for Geometric Constraint Solving

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Abstract

We use Cayley-Menger Determinants (CMDs) to obtain an intrinsic formulation of geometric constraints. First, we show that classical CMDs are very convenient to solve the Stewart platform problem. Second, issues like distances between points, distances between spheres, cocyclicity and cosphericity of points are also addressed. Third, we extend CMDs to deal with asymmetric problems. In 2*D, the following configurations are considered: 3 points and a line; 2 points and 2 lines; 3 lines. In* 3*D, we consider: 4 points and a plane; 2 points and 3 planes; 4 planes.*

Categories and Subject Descriptors (according to ACM CCS): I.3.5 [Computer Graphics]: Computational Geometry and Object Modeling

1. Introduction

The problem of solving geometric constraints often occurs in CAD, robotics, computer graphics, molecular biology, etc [Doh95, BR98]. In CAD, nowadays geometric modelers enable designers to describe geometric elements such as points, lines, circles, Bézier curves, etc in 2*D* and planes, quadrics, tori, Bézier patches, etc, in 3*D* by specifying constraints between them. Typical constraints may be: distances, angles, incidence or tangency relations. The modeler has to solve a system of constraints usually composed of polynomial equations. It decomposes the system into irreducible subsystems [HY01, GHY02], and solves them with symbolic or numerical methods such as: The Newton-Raphson iterations, homotopy-based methods, and interval analysis techniques [LM95, Dur98, HD99, Yan03]. The use of these later is less common in Geometric Constraint Solving (GCS) [JAMSR01]. However, numerical methods prevail because today symbolic packages are not powerful enough to treat 3*D* real world geometric problems.

In this paper, we show that using the cartesian coordinates to express equations of geometric constraints is neither the only nor the best approach of doing, we propose the use of Cayley-Menger Determinants (CMDs) instead. The rest of this section discusses some related works (subsection 1.1) and presents (subsection 1.2) the advantages of the intrinsic formulation: formulation independent from any particular coordinate system. Section 2 shows that CMDs are much more suitable for solving the Stewart platform problem than the usual approaches which use cartesian coordinates [Dur98, HD99]. The obtained system is much simpler, without spurious roots, and easily tractable by symbolic methods. Classical CMDs [Ber90, Hav91, Blu53] are shortly presented in section 3. New CMDs, for some asymmetric problems, are proposed for 2*D* and 3*D* examples in section 4.

1.1. Related works

Recently, D. Lesage, P. Serré and J-C. Léon, within the framework of Serré's PhD thesis [Ser00], express all 2*D* constraints in a coordinate free way [LLS02]. They don't use the Cayley-Menger formalism –which proves there are several intrinsic formulations. Instead they find independent angular and vectorial loops in some constraint graphs; then each loop gives a constraint, which is translated into equations. The unknowns are not the coordinates of points, lines, vectors, etc, but norms of vectors, and angles between vectors (which, again, are not represented by their coordinates); in other words, unknowns are scalar products between vectors. This work proves that coordinate free approaches are indeed feasible, and can be realized in a systematic way. It also proves the advantages of an intrinsic approach (see section 1.2). Podgorelec in [Pod02] and Lu Yang *et al* in [ZYY94] also propose non cartesian approaches.

Finally, to prevent a very frequent confusion, note that cartesian coordinates, Grassman Plücker coordinates, pentaspheric coordinates among others are not intrinsic formulations because each of them is dependent on a particular coordinate system.

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1.2. Advantages of the intrinsic formulation

The intrinsic formulation has several advantages [LLS02, PTRT03]. First, under this formulation some problems become tractable with symbolic computations. Second, it naturally takes into account technical unknowns and constraints (*eg*. price, temperature, strength, etc). Thus it avoids the limitations of many geometric decomposition methods. Third, the qualitative study of the resulting systems of equations is straightforward: the number of equations and unknowns are equal in correct systems; the resulting system of equations can be decomposed with bipartite graph matching methods [AAJM93], and structurally irreducible subsystems can be studied with the probabilistic numerical methods [LM98]. This contrasts with the classical cartesian formulation, where the correct systems are fixed only modulo a displacement in space; such systems are called rigid; they have less equations than unknowns (coordinates): 3 in $2D$ (2 translations and 1 rotation), 6 in $3D$ (3 translations and 3 rotations); their resolution and their geometric decomposition are thus confronted by several complications. Moreover, with the non cartesian approach, the qualitative analysis can detect mistakes often hidden with the cartesian formulation.

2. The Stewart platform problem

Given the lengths of the 12 edges of a $3D$ octahedron; the Stewart platform problem, also called the octahedron problem [Dur98, HD99, NW91], is then to find compatible coordinates for the 6 vertices $s_i, i \in [1;6]$. The 12 edges of the octahedron are:

$$s_2s_3, s_3s_4, s_4s_5, s_5s_2, s_1s_2, s_1s_3, s_1s_4, s_1s_5, s_6s_2, s_6s_3, s_6s_4, s_6s_5$$

This problem is met in CAD as a typical irreducible $3D$ problem in constraint-based geometric modeling, and in robotic with the Stewart platform: the Stewart triangular platform $s_1s_2s_3$ (Fig. 1) is driven with 6 jacks (with variable lengths) $s_1s_4, s_1s_5, s_2s_5, s_2s_6, s_3s_6, s_3s_4$ from a ground triangular base $s_4s_5s_6$. Edges of the triangular platform and of the base are rigid, *i.e.* their lengths are constant.

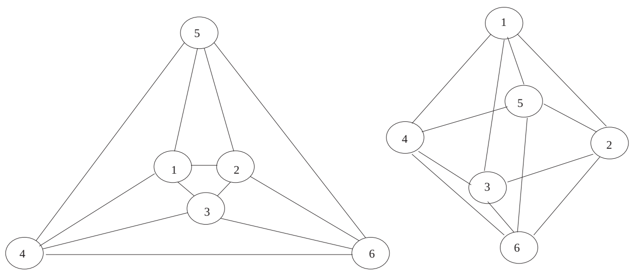

Figure 1: *Two isomorphic graphs of the Stewart platform.*

It is possible to use cartesian coordinates to pose the problem, but from the solving point of view choosing the more convenient coordinate system is not obvious, nowadays computational algebra packages are not powerful enough to solve the system, and a heavy work need to be done, by hand, in order to reduce the system to an irreducible system in 3 unknowns and 3 equations of degree 4 [Dur98, HD99]. The resulting system has Bezout number $4 \times 4 \times 4 = 64$, and BKK bound (or mixed volume) equals to 16.

Another possibility is to use CMDs. See [Ber90, Hav91] or section 3 for more details. It directly yields to 2 degree 4 equations in 2 unknowns. The CMD gives the relations between the distances between 5 points in $3D$ (between 4 points in $2D$, between $n+2$ points in nD). The CMD for the Stewart platform problem can be defined as follows:

- First, for points s_1 and s_2, s_3, s_4, s_5 (the equatorial square and the north vertex in the right part of Fig. 1). It gives an algebraic equation between squared distances
$$d_{12}, d_{13}, d_{14}, d_{15}, d_{23}, d_{24}, d_{25}, d_{34}, d_{35}, d_{45}$$
where $d_{ij} = (x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2$. All these distances are known, except d_{24} and d_{35}, the squared lengths of diagonals of the equatorial square in Fig. 1. The equation has degree 4 and involves 2 unknowns.
- Second, for points s_6 and s_2, s_3, s_4, s_5 (the equatorial square and the south vertex in the right part of Fig. 1). It gives another 4 degree algebraic equation, with the same 2 unknowns d_{24} and d_{35}. It is obvious that this equation is generically independent of the previous one.

Thus we obtain an algebraic system in 2 unknowns and 2 equations, each of degree 4. The corresponding curves can be drown in the plane using any standard curve plotting method. From the Bezout theorem, this system cannot have more than 16 solutions in complex projective space (*i.e.* taking into account multiple solutions, real and complex solutions, and solutions at infinity). Other methods yield to systems with greater Bezout number (typically 64), and in such a case it is not obvious at all to prove that there are no more than 16 solutions.

The system can be solved by any standard numerical method, say homotopy. But since there are only 2 equations in 2 unknowns, it becomes tractable with symbolic methods. For instance the Sylvester resultant, gives a degree 16 equation in only one of the unknowns. It also becomes possible to discuss degeneracies, but this question has not been investigated at this moment. Once we have the length of diagonals d_{24} and d_{35}, it is trivial to find consistent coordinates for the six vertices.

The trick here is to not to use coordinates, but to compute distances, which are independent of the coordinate system (once the scale, say meters or millimeters, has been chosen). Other parameters independent of coordinates system are angles and cross ratios, and they may be more convenient in other cases.

3. Classical Cayley-Menger determinants

This section presents an introduction to classical CMDs. See [Ber90, Hav91] for more details.

3.1. Distances between points

Given 5 points in the Euclidean $3D$ space, the following relation holds, between all their squared distances:

$$|M| = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & d_{12} & d_{13} & d_{14} & d_{15} \\ 1 & d_{21} & 0 & d_{23} & d_{24} & d_{25} \\ 1 & d_{31} & d_{32} & 0 & d_{34} & d_{35} \\ 1 & d_{41} & d_{42} & d_{43} & 0 & d_{45} \\ 1 & d_{51} & d_{52} & d_{53} & d_{54} & 0 \end{vmatrix} = 0$$

where $d_{ij} = (p_i - p_j).(p_i - p_j)$ is the square of the distance between points i and j. $|M|$ is the so called CMD.
In order to save spaces in equation writings, we define, for the rest of the paper, values $v_i = x_i^2 + y_i^2 + z_i^2$ with $i \in [1;6]$.

Proof $M = AB^t$ where A and B^t are matrices with rank at most equal to 5 (A and B have 6 rows but only 5 columns), so:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ v_1 & 2x_1 & 2y_1 & 2z_1 & 1 \\ v_2 & 2x_2 & 2y_2 & 2z_2 & 1 \\ v_3 & 2x_3 & 2y_3 & 2z_3 & 1 \\ v_4 & 2x_4 & 2y_4 & 2z_4 & 1 \\ v_5 & 2x_5 & 2y_5 & 2z_5 & 1 \end{pmatrix} \text{ and}$$

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & -x_1 & -y_1 & -z_1 & v_1 \\ 1 & -x_2 & -y_2 & -z_2 & v_2 \\ 1 & -x_3 & -y_3 & -z_3 & v_3 \\ 1 & -x_4 & -y_4 & -z_4 & v_4 \\ 1 & -x_5 & -y_5 & -z_5 & v_5 \end{pmatrix}$$

Actually, $|M|$ still vanishes when points in A and points in B are not the same. It gives another non trivial relation for distances between two point sets P_i and Q_j for $i, j \in [1;5]$ (the diagonal entries in M are no more zeros. They represent squared distances between P_i and Q_i). $\square$

The previous determinant can be extended to $2D$, $4D$, etc. Finally let us mention that the CMD is equal to a signed volume, up to some multiplicative constant.

3.2. Distances between spheres

In $3D$, one can define the signed distance (or power) of 2 spheres $S_i = (x_i \quad y_i \quad z_i)$ with radius R_i and $S_j = (x_j \quad y_j \quad z_j)$ with radius R_j as:

$$K_{ij} = K_{ji} = (x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 - (R_i^2 + R_j^2)$$

Actually, this signed distance does not depend on the system of coordinates used (once the scale is chosen, say meter or millimeter). Then the distances between any six spheres in $3D$ fulfill:

$$|K| = \begin{vmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} \\ K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66} \end{vmatrix} = 0$$

Proof $K = AB^t$ and A and B have rank at most equal to 5 (6 rows, 5 columns), so:

$$A = \begin{pmatrix} v_1 - R_1^2 & x_1 & y_1 & z_1 & 1 \\ v_2 - R_2^2 & x_2 & y_2 & z_2 & 1 \\ v_3 - R_3^2 & x_3 & y_3 & z_3 & 1 \\ v_4 - R_4^2 & x_4 & y_4 & z_4 & 1 \\ v_5 - R_5^2 & x_5 & y_5 & z_5 & 1 \\ v_6 - R_6^2 & x_6 & y_6 & z_6 & 1 \end{pmatrix} \text{ and}$$

$$B = \begin{pmatrix} 1 & -2x_1 & -2y_1 & -2z_1 & v_1 - R_1^2 \\ 1 & -2x_2 & -2y_2 & -2z_2 & v_2 - R_2^2 \\ 1 & -2x_3 & -2y_3 & -2z_3 & v_3 - R_3^2 \\ 1 & -2x_4 & -2y_4 & -2z_4 & v_4 - R_4^2 \\ 1 & -2x_5 & -2y_5 & -2z_5 & v_5 - R_5^2 \\ 1 & -2x_6 & -2y_6 & -2z_6 & v_6 - R_6^2 \end{pmatrix}$$

$\square$

This relation also holds when some radii are 0. It is then possible to compute the relation between any point and any 5 spheres in $\mathbb{R}^3$.

3.3. Cocyclicity or cosphericity of points

It is possible to express the cocyclicity of 4 points in $2D$, or the cosphericity of 5 points in $3D$, of $d+2$ points in $\mathbb{R}^d$ without coordinates, just by using squared distances between points.
In $2D$, 4 points are cocyclic (belong to the same circle) iff:

$$|C| = \begin{vmatrix} 0 & d_{12} & d_{13} & d_{14} \\ d_{21} & 0 & d_{23} & d_{24} \\ d_{31} & d_{32} & 0 & d_{34} \\ d_{41} & d_{42} & d_{43} & 0 \end{vmatrix} = 0$$

where $d_{ij} = (x_i - x_j)^2 + (y_i - y_j)^2$ is the squared distance between points i and j, and thus is independent of cartesian systems. This is equivalent to the Ptolemy theorem [Coo71].

Proof let (x_0, y_0) be the center and R_0 be the radius of the (unknown) circle. We have, in some cartesian frame (we will remove this dependency later): $(x_i - x_0)^2 + (y_i - y_0)^2 - R_0^2 = 0$ for $i = 1, 2, 3, 4$. We can express these conditions this way:

$$\begin{pmatrix} x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \\ x_4^2 + y_4^2 & x_4 & y_4 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -2x_0 \\ -2y_0 \\ x_0^2 + y_0^2 - R_0^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

It can be seen as a linear homogeneous system with unknowns in the column vector. There is a non zero solution iff the determinant of the matrix (call it C_1) is zero. We have a condition for cocyclicity, but it depends on the cartesian frame.
We can also express the system this way:

$$\begin{pmatrix} 1 & -2x_1 & -2y_1 & x_1^2 + y_1^2 \\ 1 & -2x_2 & -2y_2 & x_2^2 + y_2^2 \\ 1 & -2x_3 & -2y_3 & x_3^2 + y_3^2 \\ 1 & -2x_4 & -2y_4 & x_4^2 + y_4^2 \end{pmatrix} \begin{pmatrix} x_0^2 + y_0^2 - R_0^2 \\ x_0 \\ y_0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Here again, the determinant of the matrix (call it C_2) must vanish.
Now remark that $C = C_1 C_2^t$. Thus the determinant of $C = C_1 C_2^t$ must also vanish. We have proved the cocyclicity condition. This relation can be easily extended to $\mathbb{R}^3$ and beyond. $\square$

4. New Cayley-Menger determinants

Classical CMDs apply to very symmetric problems. Nevertheless typical problems, in CAD and constraint-based geometric modeling, are not as symmetrical as the Stewart platform problem. Constraints involve heterogeneous data: points, planes, lines, spheres ... Here are simple examples of heterogeneous CMDs.

4.1. 3 points and 1 line in $2D$

In $2D$, consider 3 points s_1, s_2, s_3 and a line l. Let d_{ij} for $i,j=1,2,3$ be the squared distances between points s_i and s_j, and let d_i be the signed (non squared) distance between point s_i and line l:$d_i = ax_i + by_i + c$. Assuming l has equation: $ax+by+c=0$ with $a^2+b^2=1$, in the cartesian frame we want to get rid of.

Due to coplanarity, there is a relation between the d_{ij} and the d_i (in passing, there is only one equality: the configuration involves 6 distances but has only five "degrees of freedom". Other possible constraints, like triangular inequalities for the triangle to be realizable; are not considered). This relation may seem a bit strange:

$$|M| = \begin{vmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & d_{12} & d_{13} & d_1 \\ 1 & d_{21} & 0 & d_{23} & d_2 \\ 1 & d_{31} & d_{32} & 0 & d_3 \\ 0 & d_1 & d_2 & d_3 & \frac{-1}{2} \end{vmatrix} = 0$$

where diagonal zeros stand for d_{ii} and $d_{ij} = d_{ji}$ of course. Note that M is symmetric despite the dissymmetry of the problem.

Proof Just check below that M is the product of the following 5×4 and 4×5 matrices(thus with rank 4, generically):

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ x_1^2+y_1^2 & 2x_1 & 2y_1 & 1 \\ x_2^2+y_2^2 & 2x_2 & 2y_2 & 1 \\ x_3^2+y_3^2 & 2x_3 & 2y_3 & 1 \\ c & -a & -b & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & -x_1 & -x_2 & -x_3 & \frac{a}{2} \\ 0 & -y_1 & -y_2 & -y_3 & \frac{b}{2} \\ 1 & x_1^2+y_1^2 & x_2^2+y_2^2 & x_3^2+y_3^2 & c \end{pmatrix}$$

□

4.2. 2 points and 2 lines in $2D$

In $2D$, consider 2 points s_1 and s_2 and 2 lines l_1 and l_2. Call $s_i l_j$ with $i,j=1,2$ the distance between point s_i and line l_j. Lines l_j have equations $a_j x + b_j y + c_j = 0$, in the cartesian frame we want to eliminate, we suppose for simplicity that $a_j^2+b_j^2=1$. Thus $s_i l_j = a_j x_i + b_j y_i + c_j$. Call $l_1 l_2$ the "distance", actually the cosine, between the 2 line directions: $l_1 l_2 = a_1 a_2 + b_1 b_2$. Call $s_1 s_2$ the squared distance between s_1 and s_2. The relation between these distances is given by the nullity of the non symmetric determinant:

$$|M| = \begin{vmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & s_1s_2 & 2s_1l_1 & 2s_1l_2 \\ 1 & s_1s_2 & 0 & 2s_2l_1 & 2s_2l_2 \\ 0 & -s_1l_1 & -s_2l_1 & 1 & l_1l_2 \\ 0 & -s_1l_2 & -s_2l_2 & l_1l_2 & 1 \end{vmatrix} = 0$$

Proof The 5×5 matrix M is the product of the following 5×4 and 4×5 matrices, which have ranks at most equal to 4, so the rank of matrix M is also at most equal to 4:

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ x_1^2+y_1^2 & 2x_1 & 2y_1 & 1 \\ x_2^2+y_2^2 & 2x_2 & 2y_2 & 1 \\ -c_1 & a_1 & b_1 & 0 \\ -c_2 & a_2 & b_2 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & -x_1 & -x_2 & a_1 & a_2 \\ 0 & -y_1 & -y_2 & b_1 & b_2 \\ 1 & x_1^2+y_1^2 & x_2^2+y_2^2 & 2c_1 & 2c_2 \end{pmatrix}$$

Note that $|M|$ is not identically zero (*i.e.* we can find entries such that $|M|$ does not vanish), since we can find in it a perfect matching (*i.e.* one generically non zero element in each and every row and column). □

4.3. 3 lines in $2D$

Let $l_i, i=1,2,3$ be any 3 lines in $2D$ having equations: $a_i x + b_i y + c_i = 0$. Assume without lose of generality that $a_i^2+b_i^2=1$. Let $c_{ij} = c_{ji} = a_i a_j + b_i b_j$ be the cosine of the angle between l_i and l_j. As well known, they fulfill:

$$\begin{vmatrix} 1 & c_{12} & c_{13} \\ c_{21} & 1 & c_{23} \\ c_{31} & c_{32} & 1 \end{vmatrix} = 0$$

$$\text{since } \begin{pmatrix} 1 & c_{12} & c_{13} \\ c_{21} & 1 & c_{23} \\ c_{31} & c_{32} & 1 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$

the ranks of these 2 last matrices is equal to 2. For $d+1$ vectors to belong to the same vectorial space of dimension d, their Gram matrix (the matrix of their scalar product [Ber90]) must have rank d, thus the determinant must vanish.

4.4. 4 points and 1 plane in $3D$

In $3D$, consider 4 points s_i, $i\in[1;4]$ and 1 plane p with equation: $ax+by+cz+d=0$, where $a^2+b^2+c^2=1$. The squared distance between two points s_i and s_j is d_{ij} and the signed distance between s_i and p is $d_i = ax_i + by_i + cz_i + d$.
This is the relation between all these distances:

$$|M| = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & d_{12} & d_{13} & d_{14} & d_1 \\ 1 & d_{21} & 0 & d_{23} & d_{24} & d_2 \\ 1 & d_{31} & d_{32} & 0 & d_{34} & d_3 \\ 1 & d_{41} & d_{42} & d_{43} & 0 & d_4 \\ 0 & d_1 & d_2 & d_3 & d_4 & \frac{-1}{2} \end{vmatrix} = 0$$

Proof

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ v_1 & 2x_1 & 2y_1 & 2z_1 & 1 \\ v_2 & 2x_2 & 2y_2 & 2z_2 & 1 \\ v_3 & 2x_3 & 2y_3 & 2z_3 & 1 \\ v_4 & 2x_4 & 2y_4 & 2z_4 & 1 \\ d & -a & -b & -c & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & -x_1 & -x_2 & -x_3 & -x_4 & a/2 \\ 0 & -y_1 & -y_2 & -y_3 & -y_4 & b/2 \\ 0 & -z_1 & -z_2 & -z_3 & -z_4 & c/2 \\ 1 & v_1 & v_2 & v_3 & v_4 & d \end{pmatrix}$$

□

4.5. 3 points and 2 planes in $3D$

Consider 3 points s_1, s_2, s_3 and 2 planes p_1 and p_2 in $3D$. Assume that p_i has equation: $a_i x + b_i y + c_i z + d_i = 0$ in some coordinate frame we want to get rid of, with $a_i^2 + b_i^2 + c_i^2 = 1$. Note $s_i p_j$ the signed distance between point s_i and plane p_j: $s_i p_j = a_j x_i + b_j y_i + c_j z_i + d_j$, and note $p_i p_j$ the cosine of the angle between p_i and p_j: $p_i p_j = a_i a_j + b_i b_j + c_i c_j$.
This is the relation between all these distances:

$$|M| = \begin{vmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & s_1s_2 & s_1s_3 & 2s_1p_1 & 2s_1p_2 \\ 1 & s_1s_2 & 0 & s_2s_3 & 2s_2p_1 & 2s_2p_2 \\ 1 & s_1s_3 & s_2s_3 & 0 & 2s_3p_1 & 2s_3p_2 \\ 0 & -s_1p_1 & -s_2p_1 & -s_3p_1 & 1 & p_1p_2 \\ 0 & -s_1p_2 & -s_2p_2 & -s_3p_2 & p_1p_2 & 1 \end{vmatrix} = 0$$

Proof

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ v_1 & 2x_1 & 2y_1 & 2z_1 & 1 \\ v_2 & 2x_2 & 2y_2 & 2z_2 & 1 \\ v_3 & 2x_3 & 2y_3 & 2z_3 & 1 \\ -d_1 & a_1 & b_1 & c_1 & 0 \\ -d_2 & a_2 & b_2 & c_2 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & -x_1 & -x_2 & -x_3 & a_1 & a_2 \\ 0 & -y_1 & -y_2 & -y_3 & b_1 & b_2 \\ 0 & -z_1 & -z_2 & -z_3 & c_1 & c_2 \\ 1 & v_1 & v_2 & v_3 & 2d_1 & 2d_2 \end{pmatrix}$$

□

4.6. 2 points and 3 planes in $3D$

In the same way as above, distances between 2 points s_1, s_2 and 3 planes p_1, p_2, p_3 in $3D$ are linked by the following relation:

$$|M| = \begin{vmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & s_1s_2 & 2s_1p_1 & 2s_1p_2 & 2s_1p_3 \\ 1 & s_1s_2 & 0 & 2s_2p_1 & 2s_2p_2 & 2s_2p_3 \\ 0 & -s_1p_1 & -s_2p_1 & 1 & p_1p_2 & p_1p_3 \\ 0 & -s_1p_2 & -s_2p_2 & p_1p_2 & 1 & p_2p_3 \\ 0 & -s_1p_3 & -s_2p_3 & p_1p_3 & p_2p_3 & 1 \end{vmatrix} = 0$$

Proof

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ v_1 & 2x_1 & 2y_1 & 2z_1 & 1 \\ v_2 & 2x_2 & 2y_2 & 2z_2 & 1 \\ -d_1 & a_1 & b_1 & c_1 & 0 \\ -d_2 & a_2 & b_2 & c_2 & 0 \\ -d_3 & a_3 & b_3 & c_3 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & -x_1 & -x_2 & a_1 & a_2 & a_3 \\ 0 & -y_1 & -y_2 & b_1 & b_2 & b_3 \\ 0 & -z_1 & -z_2 & c_1 & c_2 & c_3 \\ 1 & v_1 & v_2 & 2d_1 & 2d_2 & 2d_3 \end{pmatrix}$$

□

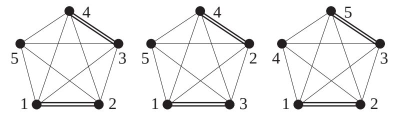

Figure 2: *Isomorphic subgraphs of the same class monomials*

4.7. 4 planes in $3D$

Like in section 4.3, we only need to make the determinant of the Gram matrix of 4 plane normals in $3D$ vanish.

5. Future extensions

A first open problem is to find relations involving also lines in $3D$, and not only points and planes. May be Grassman Plücker coordinates for lines in some cartesian frame must be used, before the frame elimination. One such relation, due to Neil White, is given in Sturmfels's book [Stu93], th. 3.4.7: it is the condition for five lines in $3D$ space to have a common transversal line. Philippe Serré, in his PhD thesis [Ser00], also gives the relation involving distances between two lines AB and CD and between points A, B, C, D.

A second problem is to find such polynomial relations. From a theoretical point of view, it suffices to use a Grobner package to eliminate variables representing coordinates in some set of equations (for instance equations: $(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 - d_{ij}^2 = 0, i \in [1;4], j \in [i+1;5]$, to find the Cayley-Menger equation relating distances between 5 points in $3D$). In practice, Grobner packages are not powerful enough. The polynomial condition can be computed by interpolation: for instance, to guess the Cayley-Menger equation in $3D$, one can proceed in three steps:

- Generate N random configurations of 5 points $(x_i, y_i, z_i) \in \mathbb{Z}^3$,
- Compute square distances $d_{ij}^k, i \in [1;4]$ and $j \in [i+1;5]$ for each configuration $k \in [1;N]$. This gives N 15D points.
- All these N 15D points lie on the zero-set of an unknown polynomial in the variables d_{ij}. We search for this polynomial by trying increasing degrees.

This polynomial has an exponential number of monomials, so there is an exponential number of unknown coefficients. However, due to the symmetry some monomials have the same coefficients and are said to lie in the same "class". For instance monomials $d_{12}^2 d_{34}^2$,

$d_{13}^2 d_{24}^2$, etc lie in the same class: Monomials of the same class correspond to isomorphic edge weighted subgraphs of K_5, the complete graph with 5 vertices and with edges weighted by the degree of the corresponding monomial (Fig. 2). To be feasible this approach must exploit this symmetry to reduce the number of unknown coefficients to the number of classes. The fast generation of these classes (and of one instance per class) is an interesting and non trivial combinatorial problem by itself, related to the Polya's counting theory.
To validate this approach we implemented a simple algorithm that computes Cayley-Menger relations and distance relations for 6 2D points to lie on the same conic as well as for 10 3D points to lie on the same quadric. We noticed that this first implementation works slowly because it doesn't exploit the symmetry. Moreover its output (the polynomial coefficients) has an exponential size and is thus unusable. Exploiting symmetry is thus essential.

6. Conclusion

This paper has shown that CMDs may give simpler algebraic systems, with less spurious roots, and tractable with today's symbolic algebra packages. Examples of points/points, circles/circles and spheres/spheres relations are given. Unfortunately, these classical CMDs involve only relations between geometric primitives of the same type. This paper also introduced new CMDs formulations to find relations between heterogeneous $2D$ and $3D$ geometric entities. It remains to propose an automatic method to generate such new Cayley-Menger relations: a new challenging problem for computational algebra.

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