Some Lemmas to Hopefully Enable Search Methods to Find Short and Human Readable Proofs for Incidence Theorems of Projective Geometry

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Abstract. Search methods provide short and human readable proofs, *i.e.* with few algebra, of most of the theorems of the Euclidean plane. They are less succesful and convincing for incidence theorems of projective geometry, which has received less attention up to now. This is due to the fact that basic notions, like angles and distances, which are relevant for Euclidean geometry, are no more relevant for projective geometry. This article suggests that search methods can also provide short and human readable proofs of incidence theorems of projective geometry with well chosen notions, rules or lemmas. This article proposes such lemmas, and show that they indeed permit to find by hand short proofs of some theorems of projective geometry.

1 Introduction

What is a proof? In a first acceptation, a proof is a guarantee, a certificate that some theorem holds; these proofs are very detailed and rigorous, *e.g.* they account for degenerate cases. These proofs are not intended to be read or understood by a human: they can be tedious computations, resorting to some algorithms in Computer Algebra (Wu-Ritt method, Gröbner bases); the guarantee is due to the correctness of the Computer Algebra program. Such proofs provide certitudes, but do not always bring understanding or enlightenment. In a second acceptation, a proof brings us (*i.e.* human beings) explanation, knowledge, understanding, and even enlightenment: *e.g.* these proofs may suggest generalizations, or apply to more general theorems. They must be easy to read and understand. The shorter the proof, the better. Visual proofs are extreme examples of such proofs. Details such as degeneracies must not occlude the main arguments of these proofs.

This paper considers the possibility for search methods to produce short and human readable proofs (*i.e.* with as few algebra as possible) for theorems of projective geometry, mainly for the projective plane. The hypothesis and the conclusion of these theorems are point-line or point-conic incidences. An avenue to compute short proofs is to apply powerful lemmas, in contrast to proofs of the first kind which relies on long and tedious algebraic computations.

Previous works: search methods [7, 8, 6, 2, 3, 5, 17] give human readable proofs in Euclidean geometry with very few algebra. Many consider a typical figure to prune the search combinatorial space and discard irrelevant degenerate cases in the wake of 1959 Gelertner's pioneering work [7, 8, 6]. However the ruleset of current search methods is not well suited to prove incidence theorems of projective geometry; for example angles, distances, similitudes, isometries are relevant for Euclidean geometry, but not for projective geometry. Raymond Pouzergues [13] (unfortunately in French only) proves by hand 2 dozens of incidence theorems in the projective plane, relying on a variant of Pascal's Mystical Hexagram theorem as a main lemma, which he calls the hexamys theorem. Michelucci and Schreck [11] automatize the search of hexamys. But they do not rely on a typical figure to prune the combinatorial space and to discard irrelevant degenerate cases, and (with hindsight) their ruleset is not powerful as it could be; for instance they do not use brianchons (hexamys duals, defined below). Richter-Gebert et al proposed combinatorial-algebraic proofs [1, 14] called binomial proofs.

This article proposes lemmas and rules which could make search methods able to provide short and human readable proofs of incidence theorems of projective geometry.

Only some of the proofs given below have been found after a computer search (with an ad-hoc program). The proofs given below must be considered as an empirical evidence that the rules or lemmas which are proposed indeed permit to find short and human readable proofs of incidence geometry. This article does not focus on the algorithmic part of combinatorial search methods. I will only mention that a feature of theorems in projective geometry, and an issue (or an opportunity?), is their big number of symmetries.

The plane of this article is as follows: §2 presents the main lemmas usable in short proofs. Then §3 proves Desargue's theorem, §4 proves Desargue's theorem in the Cevian case, §5 proves the 3 chords theorem and a generalization, §6 explicits the dual of this theorem, §7 proves the 3 circles theorem and its generlization, §8 proves the 4 circles theorem and a generalization. §9 gives some algorithmics to compute automatically this kind of proofs. §10 concludes.

2 Chasles, Pascal, Brianchon

Powerful lemmas enable short proofs. A main lemma which seems able to prove a significant number of theorems involves cubic curves; it was first proved by Michel Chasles and later generalized to curves of higher degree by Cayley and Bacharach.

Lemma 1. *In the projective complex plane, all cubic curves C which pass through 8 of the 9 (distinct) intersection points of 2 other cubic curves C_1 and C_2 (without common component) also pass through the 9th point.*

This theorem solves an apparent contradiction. On one hand, a cubic curve is defined by 9 different points, under some genericity conditions (for instance, no 4 of the points lie on a common line, and no 7 of the points lie on a common conic). On the second hand, after Bézout theorem, two cubic curves (without common component) C_1 and C_2 in the complex projective plane intersect in exactly 9 different points (in the generic case); but these 9 intersection points do not define an unique cubic curve, because the two cubic curves C_1 and C_2 (and all linear combinations $C(x, y, h) = tC_1(x, y, h) + (1 - t)C_2(x, y, h)$ where $C_i(x, y, h) = 0$ is the homogeneous equation of the curve C_i) are different and pass through the 9 points. The solution to this apparent dilemma is to realize than the 9 intersection points are not independent. Actually they have rank 8, in a sense precised in the following proof of Chasles' theorem:

Proof. Assume points have homogeneous coordinates (x, y, h) in $\mathbb{C}^3 \setminus (0, 0, 0)$. Define $\phi : \mathbb{C}^3 \to \mathbb{C}^{10}$,

$$\phi(x, y, h) = (x^3, y^3, h^3, x^2y, x^2h, y^2h, xy^2, xh^2, yh^2, xyh)$$

Then every cubic curve has equation $Q \cdot \phi(x, y, h) = 0$, where $\cdot$ denotes the Hermitian scalar product, and Q is a non zero vector in $\mathbb{C}^{10}$. Each cubic curve is represented with an hyperplane in $\mathbb{C}^{10}$. Hyperplanes in $\mathbb{C}^{10}$ have rank 9. The intersection of 2 hyperplanes (representing the intersection of 2 cubic curves) has rank 8. Now, after Bézout' theorem, two cubic curves intersect in 9 different points in generic case. Thus the 9 intersection points $\phi(p_1), \phi(p_2), \ldots \phi(p_9)$ between the 2 cubics have rank 8: only 8 of the 9 points are independent, and the 9th lies in the vector space spanned by the 8 others.

Remark 1. Rank 10 matroids capture Chasles' theorem. A method to prove incidence theorems searches the matroids compatible with the hypothesis incidences [11].

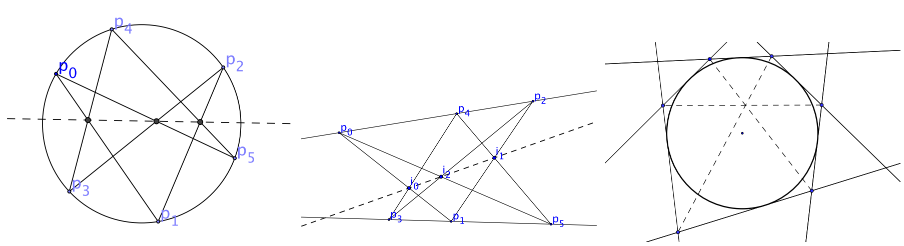

Fig. 1. From left to right: Pascal', Pappus', Brianchon's theorems.

Chasles' theorem permit to prove the Pascal mystical hexagram theorem (Fig.1):

Theorem 1 (Pascal's mystical hexagram). *The opposite sides of an hexagon inscribed in a conic curve meet in 3 colinear points.*

Proof (with Chasles theorem). Let $p_0, p_1, \dots p_5$ the 6 points on a conic. The 3 intersection points of opposite sides are $i_0 = p_0p_1 \cap p_3p_4$, $i_1 = p_1p_2 \cap p_4p_5$ and $i_2 = p_2p_3 \cap p_5p_0$. Call C_1 the cubic curve which is the union of the 3 lines p_0p_1, p_2p_3 and p_4p_5. Call C_2 the cubic curve which is the union of the 3 lines p_1p_2, p_3p_4 and p_5p_0. C_1 and C_2 meet at the 9 intersection points $p_0, \dots p_5, i_0, i_1, i_2$. The cubic curve C is the union of the conic curve through the p_is and of the line i_0i_1. C passes through 8 of the 9 points (namely the p_is and i_0 and i_1). Thus after Chasles' theorem, C also passes through the 9th point i_2. Admitting i_2 does not lie on the conic (an example, *i.e.* a figure –also called a witness– is a visual proof sufficient and very convenient for a human), i_2 must lie on the line i_0i_1.

Chasles' theorem permits to prove Pappus' theorem (Fig.1):

Theorem 2 (Pappus). *3 points p_0, p_2, p_4 lie on a first line, and 3 points p_1, p_3, p_5 lie on a second line. Then the 3 intersection points $i_0 = p_0p_1 \cap p_3p_4, i_1 = p_1p_2 \cap p_4p_5, i_2 = p_2p_3 \cap p_5p_0$ are colinear.*

Proof (with Chasles theorem). Define C_1 and C_2 as before: C_1 is the cubic curve which is the union of the 3 lines p_0p_1, p_2p_3 and p_4p_5. C_2 is the cubic curve which is the union of the 3 lines p_1p2, p_3p_4 and p_5p_0. C_1 and C_2 meet at the 9 intersection points $p_0, \dots p_5, i_0, i_1, i_2$. The cubic curve C is the union of the line $p_0p_2p_4$, the line $p_1p_2p_3$, and the line i_0i_1. C passes through 8 of the 9 points, thus it passes through the 9th point which is i_2. Admitting i_2 does not lie on the lines $p_0p_2p_4$ nor $p_1p_3p_5$ (an example, *i.e.* a figure, is sufficient), i_2 must lie on the line i_0i_1.

Remark 2. This line of thought was introduced by Chasles. It has been somewhat forgotten for the benefit of Bourbaki style. It is today revisited, for instance in Richter-Gebert's book [15].

Pouzergues reformulates Pascal' theorem as follows:

Definition 1. *An hexamys is an hexagon $p_0p_1p_2p_3p_4p_5$ such that opposite sides meet in 3 colinear points (either 3 distinct colinear point, or 2 distinct points) $i_0 = p_0p_1 \cap p_3p_4$, $i_1 = p_1p_2 \cap p_4p_5$ and $i_2 = p_2p_3 \cap p_5p_0$.*

Theorem 3 (Hexamys). *All permutations of an hexamys are hexamys.*

Proof. Trivially, the 6 points of an hexamys lie on a conic, whatever the permutation of the 6 points.

Pouzergues [13], then Michelucci and Schreck [11], use hexamys to prove incidence theorems in the projective plane: a colinearity between 3 points i_0, i_1, i_2 (together with 6 lines: d_0, d_0' through i_0, d_1, d_1' through i_1, d_2, d_2' through i_2) generates an hexamys, every permutation of which imply new colinearities. Hexamys also permit to prove concurrences of 3 lines.

Instead or together with hexamys, it is possible to use Brianchons, from Brianchon's theorem. Brianchons permits to prove concurrence of lines. Brianchon's theorem (Fig.1) states that

Theorem 4 (Brianchon). *If a conic is inscribed in an hexagon with vertices* $p_0p_1p_2p_3p_4p_5$ *(i.e. the 6 lines* p_0p_1*, ...* p_4p_5*,* p_5p_0 *of the hexagon are tangent to the conic) then the 3 diagonal lines of the hexagon, namely* p_0p_3, p_1p_4, p_2p_5*, are concurrent.*

Proof. with Chasles. Omitted for conciseness.

It is possible to cancel all references to conics in Brianchon's theorem, as we do for Pascal's.

Definition 2 (brianchon). *A brianchon is an hexagon with lines* $d_0d_1d_2d_3d_4d_5$ *and vertices* $p_i = d_i \cap d_{(i+1) mod\ 6}$ *and such that the 3 diagonal lines* p_0p_3*,* p_1p_4*,* p_2p_5 *are concurrent.*

Brianchon's theorem can be restated as:

Theorem 5. *Every permutation of the lines of a brianchon is a brianchon.*

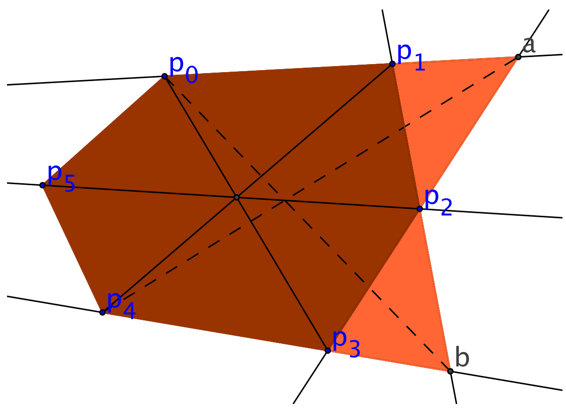

Fig. 2. Brianchon's theorem: if $p_0, p_1, p_2, p_3, p_4, p_5$ is a brianchon, then p_0, $a = p_0p_1 \cap p_2p_3, p_2$, $b = p_1p_2 \cap p_3p_4$, p_4, p_5 is a brianchon as well.

It suffices to prove this theorem for a transposition (an exchange), since transpositions generate the group of permutations (Fig.2).

Proof. with Pappus. A brianchon has vertices p_0, p_1, p_2, p_3, p_4, p_5 and lines $d_0 = p_0p_1, \dots d_5 = p_5p_0$. Let us exchange lines d_1 and d_2, and prove that the hexagon with lines d_0, d_2, d_1, d_3, d_4, d_5, and with vertices p_0, p_1, $a = d_0 \cap d_2 = p_0p_1 \cap p_2p_3$, p_2, $b = d_1 \cap d_3 = p_1p_2 \cap p_3p_4$, p_4, p_5 is a brianchon. So we need to prove that the 3 diagonal lines p_0b, ap_4, p_2p_5 are concurrent. By hypothesis, p_0p_3, p_1p_4, p_2p_5 concur in some point o. Apply Pappus' theorem on the 3 colinear points: p_0, p_1, a and on the 3 colinear points p_4, p_3, b; it implies that the 3 points:

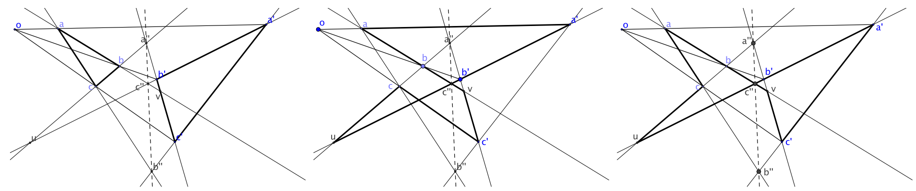

Fig. 3. The 2 triangles in perspective of Desargue's theorem; the first hexamys with points o, b', b colinear by hypothesis; the second hexamys which proves that b'', a'', c'' are colinear.

$p_0p_3 \cap p_1p_4 = o, p_1b \cap ap_3 = p_2, ap_4 \cap p_0b = x$ are colinear. Thus the point x lies on ap_4, on p_0b and on $op_2 = p_5p_2$. Thus the hexagon with vertices p_0, a, p_2, b, p_4, p_5 is a brianchon.

Proof. by hexamys. Omitted for conciseness.

Proof. by Chasles. In the previous proof by Pappus' theorem, replace Pappus' theorem with its proof by Chasles.

Remark 3. A combinatorial search for brianchons (find 3 concurrent lines, and 2 points on each line) in a specified configuration permits to deduce new brianchons, and thus new triples of concurrent lines. It also permits to prove colinearities.

Another short proof of Brianchon's theorem is

Proof. By duality: Brianchon's theorem is the dual of Pascal's theorem.

Indeed duality is another powerful lemma which yields short proofs. Duality exchanges the roles of points and lines, preserving incidences. Gergonne [4] realized first that all the theorems in the projective plane can be dualized. Duality exchanges circles (conics, cubics) with dual circles (conics, cubics). A dual circle (conic, cubic) is a set of lines tangent to a circle (conic, cubic). Duality is used in §6.

3 Desargue's theorem

Desargue's theorem (Fig. 3) is a combinatorial property of 5 planes in 3D, which still holds after projection on any plane:

Theorem 6 (Desargue theorem.). *Let a, b, c and a', b', c' be 2 triangles in perspective, i.e. the 3 lines aa', bb', cc' concur in a point o. Then the 3 intersection points between homologous sides: $c'' = ab \cap a'b'$, $a'' = bc \cap b'c'$, and $b'' = ca \cap c'a'$ are colinear.*

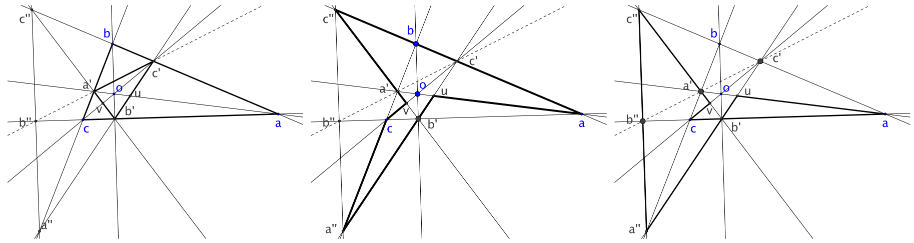

Fig. 4. Left to right: the 2 triangles in perspective in the Cevian case (a' lies on bc, etc); c, a'', u, a, c'', v is an hexamys because opposite sides cross in 3 points b, b', o colinear by hypothesis; thus c, a, u, a'', c'', v is also an hexamys; thus b'', a', c' are colinear.

Proof (with dimension lifting). Assume the triangles abc and $a'b'c'$ lie in 2 distinct planes, in 3D, and are still in perspective when viewed from point o. Points a, b, a', b' are coplanar (since lines aa' and bb' cross at o). Thus lines ab and $a'b'$ are coplanar and intersect at some point c'' (possibly at infinity). Now, line ab lies on plane abc, line $a'b'$ lies on plane $a'b'c'$, thus these 2 lines must intersect somewhere along the intersection line l of planes abc and $a'b'c'$. The same holds for a'' and b'': they lie on l (here we use symmetry to factorize and shorten the proof). Thus a'', b'', c'' are colinear.

Remark 4. This proof is captured by rank 4 matroids. This kind of proof is used in [9] (in [16]).

Proof (By hexamys (thus by Chasles)). Define $u = a'b' \cap bc$ and $v = ab \cap b'c'$. (a, a', u, c, c', v) is an hexamys because its opposite sides meet in points o, b', b, aligned by hypothesis. Thus (a, c, u, a', c', v) is another hexamys, the opposite sides of which meet in 3 aligned points: b'', a'', c''.

4 Desargue's in Cevian case

In the cevian case of Desargue's theorem (Fig.4), the two triangles are still in perspective, but the vertices of one triangle lie on the edges of the second triangle.

Theorem 7 (Desargue in Cevian case). *Again, 2 triangles abc and $a'b'c'$ are in perspective viewed from point o. Moreover each of the vertices a', b', c' lies on the corresponding side bc, ca, ab. As in the generic case, homologous sides intersect at colinear points $a'' = bc \cap b'c'$, $b'' = ca \cap c'a'$, $c'' = ab \cap a'b'$.*

Proof (with hexamys). The following proof (see Fig.4) needs only one hexamys and is much simpler than the proof in [11]. It was found with a computer search. Points a, b, c, o are given. As usual define $a' = oa \cap bc$, $b' = ob \cap ac$, $c' = oc \cap ab$. Then define $a'' = bc \cap b'c'$, $c'' = ab \cap a'b'$, and here comes the unusual thing: $b'' = a''c'' \cap ac$; thus a'', b'', c'' are colinear but we have now to prove that b'' indeed lies on $a'c'$. Pose $u = oa \cap b'c'$, and $v = oc \cap a'b'$. Then c, a'', u, a, c'', v is

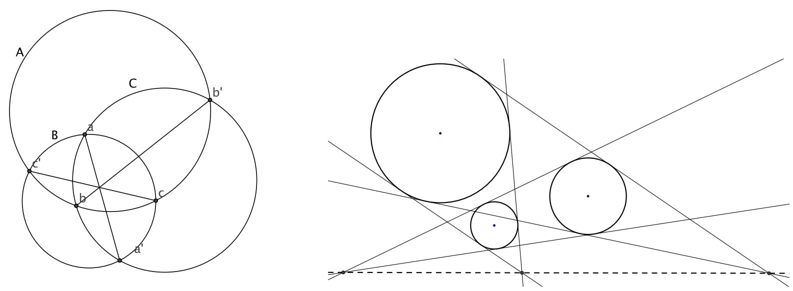

Fig. 5. Left: The 3 pairwise common chords concur. Right: the dual theorem: The 3 homothety centres of pairwise circles are colinear. For readibility, only 3 centres and 1 line are displayed. Actually there are 6 centres, forming 4 lines.

an hexamys because its opposite sides intersect in b, b', o colinear by hypothesis; thus after permutation, c, a, u, a'', c'', v is also an hexamys; its opposite sides intersect at points b'', a', c', thus b'' indeed lies on line $a'c'$.

5 The 3 chords theorem

Theorem 8 (The 3 chords theorem). *Let A, B, C be 3 intersecting circles. Apart cyclic points, A and B meet in points c, c', A and C meet in points b, b', B and C meet in points a, a'. Then the 3 chord lines aa', bb', cc' concur.*

Remark 5. Circles are objects living in the Euclidean plane, not in the projective plane. But we will replace circles by conic in a moment.

Proof. By Chasles. The cubic curve A' is the union of circle A and line aa'. The cubic curve B' is the union of circle B and line bb'. The cubic curve C' is the union of circle C and line cc'. The 2 cubic curves A' and B' intersect in 9 points: $a, b, c, a', b', c', I, J, aa' \cap cc' = o$, where I and J are the two cyclic points (they have homogeneous coordinates $(1, \pm\sqrt{-1}, 0)$ and belong to all circles). The cubic curve C' passes through the first 8 of these points. By Chasles' theorem, $C' = C \cup (cc')$ passes also through the 9th point $aa' \cap cc' = o$. Since o' does not lie on C (a witness, *i.e.* a figure, is a sufficient visual proof for a human), o' lies on line cc'. Thus the 3 chords are concurrent.

The usual proof is as follows: first the power of a point p relatively to a circle C is defined; let l an arbitrary line through p which cuts C in points c and c'. Then the power of p relatively to C is the product $(\bar{c} - \bar{p})(\bar{c'} - \bar{p})$ where $\bar{c}, \bar{c'}, \bar{p}$ are abscissas of points c, c', p along the line. Then it is proved that the power is independent on the line l, and that the line of the common chord of two circles is the locus of points with equal power relatively to the 2 circles. Finally, if o lies on the common chords of circle A and B, and of A and C, then o has equal power relatively to circles A, B and C, thus o lies on the third common chord of

B and C. The proof is partly algebraic, but short enough to be human readable. But it is hard to generalize this theorem. The proof by Chasles proves more than this theorem. Actually, the proof by Chasles' theorem proves the more general theorem:

Theorem 9. *Let A, B, C be 3 conics. All 3 conics pass through 2 common distinct points (called I and J in the initial 3 chords theorem). A and B also intersect in c and c', B and C also intersect in a, a', and A and C intersect in b, b'. Then after the previous Chasles' proof, the lines aa', bb' and cc' concur.*

We mention yet another proof of the 3 chords theorem: Chasles' theorem lifts in dimension 10, but dimension 3 is sufficient (and more intuitive):

Proof. Dimension lifting. Lift the Euclidean plane on the parabolic sheet $z = x^2 + y^2$: $L(x, y) = (x, y, z = x^2 + y^2)$. Cocyclic points in the plane become coplanar points after lifting. The common chord of 2 circles A and B is the projection on the plane Oxy of the intersection line between the 2 planes $L(A)$, $L(B)$ of the lifted circles. Now, the 3 planes of $L(A), L(B), L(C)$ in 3D intersect in one common point.

6 The dual of 3 chords theorem

Duality is illustrated with the dual of the 3 chords theorem, in Fig 5. Both theorems and their proofs can be dualized.

Theorem 10 (Dual of the 3 chords theorem.). *Let A, B, C be 3 circles. Lines a, a' are common tangents to B and C, Lines b, b' are common tangents to A and C. Lines c, c' are common tangents to A and B. Then the 3 intersection points $a \cap a', b \cap b', c \cap c'$ are colinear.*

Proof (Usual proof). $a \cap a'$, etc is the centre of the scaling (homothety, or homothecy, a non-rotating dilation) which maps circle B to C. This scaling is equal to the composition of the scaling which maps circles B to A (with centre $c \cap c'$), and the scaling which maps circles A to C (with centre $b \cap b'$). These 2 scalings leave globally invariant the line joining their centres $c \cap c'$ and $b \cap b'$. Thus $c \cap c'$ lies on this line, using the lemma: lines globally invariant through a scaling all pass through the centre of the scaling.

For conciseness, the dualization of other theorems (Chasles', theorem 9, etc) and their proofs are left to the reader.

7 The 3 circles theorem

Theorem 11. *The 3 circles theorem. Let a, b, c be the 3 vertices of a triangle. Let a' be any point on line bc, let b' be any point on line ac, and c' any point on line ab. Let A be the circle through points a, b', c', let B be the circle through points b, a', c', and C be the circle through points c, a', b'. Then the 3 circles A, B, C have another common point ω.*

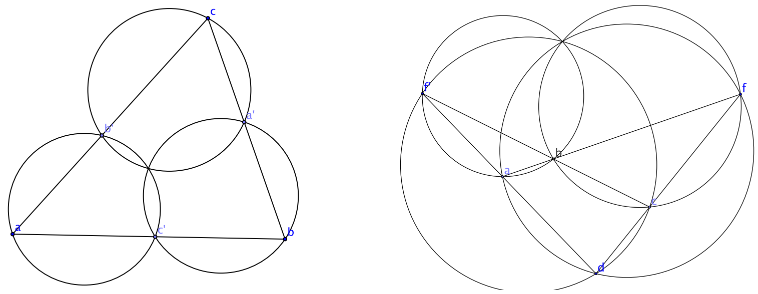

Fig. 6. Left: Three circles theorem: the 3 circles share a common point (other than the 2 cyclic points). Right: the 4 circles theorem, the four circles share a common point.

Proof. by Chasles. See Fig.6. Let A' be the cubic which is the union of circle A and line $a'bc$, B' the cubic which is the union of circle B and line $ab'c$, and C' the cubic which is the union of circle C and line abc'. The 2 cubic curves A' and B' meet in 9 different points $a, b, c, a', b', c', I, J, \omega$ where I, J are the 2 cyclic points common to all circles, and ω is the intersection point of $A \cap B$ which is not c'. The third cubic C' passes through the first 8 of these 9 points. Thus after Chasles' theorem, it also passes through the 9th point ω. Thus the 3 circles share a common point, ω.

Remark 6. Again, the proof by Chasles' theorem proves more than the 3 circles theorem, because the cyclic points I and J can be replaced with any generic points. It proves the following theorem:

Theorem 12 (A triangle and 3 conics). *Let a, b, c be 3 points, let a', b', c' be 3 points with $a' \in bc, b' \in ac, c' \in ab$. Let I, J be 2 generic distinct points which do not lie on lines ab, ac, bc. Let A be the conic curve through 5 points a, b', c', I, J, let B' be the conic curve through 5 points b, a', c', I, J, let C' be the conic curve through 5 points c, a', b', I, J. Then the 3 conics share another intersection point ω.*

8 The 4 circles theorem

Theorem 13 (The 4 circles theorem.). *Let a, b, c, d be 4 points in generic position. Let $f = ab \cap cd$, and $f' = ac \cap bd$. Let C_{ab} be the circle through a, b, f; let C_{cd} be the circle through a, b, f; let C_{bc} be the circle through b, c, f'; let C_{ad} be the circle through a, d, f'. Then the 4 circles $C_{ab}, C_{cd}, C_{bc}, C_{ad}$ share another common point, which is not a cyclic point.*

Proof (by Chasles). See Fig.6. Let C'_{ab} be the cubic curve which is the union of C_{ab} and line cdf'; let C'_{cd} be the cubic curve which is the union of C_{cd} and line abf'; let C'_{bc} be the cubic curve which is the union of C_{bc} and line adf; let C'_{ad}

be the cubic curve which is the union of C_{ad} and line bcf; then the 2 cubics C'_{ab} and C'_{cd} intersect in 9 distinct points $a, b, c, d, f, f', I, J, \omega$, where I, J are the two cyclic points common to all circles, and ω is the other intersection point of circles C_{ab} and C_{cd} (the 3 other intersection points are f and I, J). The cubic curve C'_{bc} passes through the 8 first of these 9 points, so after Chasles' theorem, it also passes through the 9th point, ω. Since ω does not lie on the line adf, component of the cubic curve C'_{bc} (a figure or witness is a sufficient visual proof), it means that ω lies on the other component of C'_{bc}, the circle C_{bc}. Similarly for the cubic C'_{ad}, which is left to the reader (A symmetry argument, in fact a permutation, can also be used).

Remark 7. Again, Chasles' proof proves more: I and J can be generalized to any (generic) points.

Theorem 14. *Let a, b, c, d, i, j be any generic points and $f = ad \cap bc$, $f' = ab \cap cd$. Points i and j generalize previous cyclic points I and J, they are any point (i generic position). The 4 conics $C_{ab}, C_{cd}, C_{bc}, C_{ad}$ share points i and j. Moreover the conic C_{ab} passes through a, b, f, the conic C_{cd} passes through c, d, f, the conic C_{bc} passes through b, c, f', the conic C_{ad} passes through a, d, f'. Then the 4 conics share another common point ω.*

Proof. In the previous proof by Chasles, replace I with i, and replace J with j.

9 Automatization

All previous proofs share the same combinatorial flavor and resort to the same lemmas arguments (Pascal', Chasles', Brianchon's theorems), which suggests that the search of such proofs can be automatized with search methods [11]. It will extend the naive algorithm in [11]: it also considers brianchons, and it relies on a witness, *i.e.* it considers a typical figure to prune the combinatorial search space and discard irrelevant degenerate cases.

In a nutshell, users provide (possibly interactively) the hypothesis and the conclusion of a conjecture. Hypothesis and conclusion involve only incidences. All incidences (point-circle incidences, point-conic incidences, point cubic incidences) are internally reducible to point-line incidences: a circle is just a conic passing through two constant points (the cyclic points), 6 points on the same conic are an hexamys, and a cubic is the union of 3 distinct lines, or of a line and a proper conic.

Users also provide a witness. A witness is a figure, which illustrates the conjecture to be proved, and where vertices (and possibly lines and conics) have numerical coordinates (either rational, floating-point, interval), and names. H. Gelernter is the first to rely on a witness to discover and prove geometric theorems in 1959 [7, 8, 6, 5]. More recently, witnesses are used to detect dependences in systems of geometric constraints, and to decompose and solve systems of geometric constraints [10, 12].

The witness first permits to check that the user makes no mistake when specifying the hypothesis and the conclusion: the witness must satisfy the conjecture (otherwise the conjecture has a counterexample, or more likely, the user makes some mistake when specifying the problem). Also, when completing the figure with (typically) intersection points between lines, the witness is used to check that created intersection points are indeed new (different from the vertices) and all distinct. Note that when two intersecting points, or a vertex and an intersection point are equal in the witness, it provides a conjecture, which the user may try to prove, but not a fact. Conversely, when two intersecting points, or a vertex and an intersection point, are numerically different[1], this is considered as a fact, and the witness is considered as a proof. In passing, we tried to prove non colinearities and non concurrences with logic and some matroid rules, but the computations are slow and the obtained proofs are long, tedious and boring; the visual proof provided by the witness is the best in all aspects.

The proof searcher (”proof assistant” would be confusing) provides several tools. One tool is a combinatorial and straightforward search of hexamys (as in [11]) and brianchons which prove the conjecture. It is also possible to search to apply Pappus' or Desargue's theorems.

When this search fails, users have two non exclusive possibilities: first, they can ask the proof searcher to complete the figure with intersection points between two lines (or conics) of the figure, or with lines joining two vertices; we already underlined the essential role played by the witness during the completion (the previous method [11] used no witness, which is its main weakness: degeneracies could not be handled). Second they can ask the prover to search for other conjectures, *i.e.* other colinearities of 3 points or concurrences of 3 lines, which are not specified in the hypothesis, but which are (numerically, and approximately) fulfiled in the witness. The proof searcher and users then interactively try to recursively prove these conjectures. Proved conjectures are added to the hypothesis. The proof searcher then checks if these enriched hypothesis contain an hexamys or a brianchon which proves the initial conjecture. Of course, many tactics (backward chaining / forward chaining) can be imagined and implemented, in the wake of search methods [5]. Also, several classes of inner representations can be considered; for instance one may imagine to rely on matroids [11, 9], or a combination of several matroids (rank 3 for lines, rank 6 for conics, rank 10 for cubics, plus some transition rules). These questions deserve further study.

10 Conclusion

In the hope to enable current search methods [5] to find short and human readable proofs of incidence theorems in projective geometry, this article proposes some rules, *i.e.* lemmas: Chasles, Pappus, Pascal and Brianchon's theorems which may be powerful enough. For conciseness, some relevant concepts

[1] far enough from each other, say one pixel, to account for the numerical inaccuracy; this heuristic is used in Cabri, Cindarella, and other dynamic geometry softwares

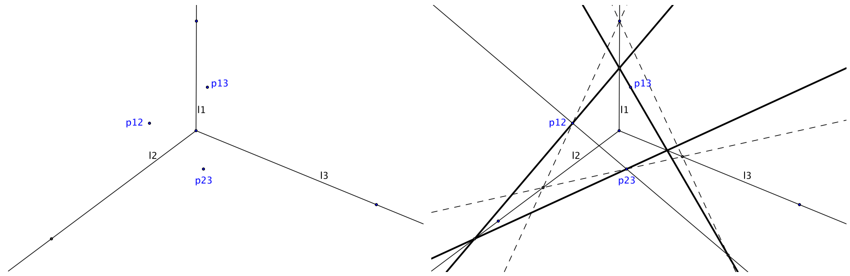

Fig. 7. Left: let l_1, l_2, l_3 be three given concurrent lines; let p_1, p_2, p_3 be 3 given points. Find 3 points $x_1 \in l_1$, $x_2 \in l2$, $x_3 \in l_3$ such that the line x_1x_2 passes through p_{12}, the line x_2x_3 passes through p_{23}, and the line x_1x_3 passes through p_{13}. Right: a construction with ruler only, which relies on Desargue' theorem.

could not be mentioned: projectivities, perspectivities, colineations, homographies, involutions, cross ratios, etc, though Coxeter [4] relies only on them to prove our lemmas, *i.e.* basic theorems of projective geometry: Pappus', Pascal', Brianchon's theorems, etc. Proofs à la Coxeter should also be considered and computed, and compared with proofs proposed in this article.

Finally, short and human readable proofs should permit to automatically extract, and prove geometric constructions with ruler and compass, or with ruler alone, for incidence problems like: solving the problem in Fig. 7, constructing the intersection points of a given line and a conic given by 5 points, constructing with the ruler only the second intersection point when the first is known, etc (see Cabri web pages for solutions).

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