What Is a Line ?

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Abstract. The playground is the projective complex plane. The article shows that usual, naive, lines are not all lines. From naive lines (level 0), Pappus geometry creates new geometric objects (circles or conics) which can also be considered as (level 1) lines, in the sense that they fulfil Pappus axioms for lines. But Pappus theory also applies to these new lines. A formalization of Pappus geometry should enable to automatize these generalizations of lines.

1 Introduction: What Is a Line ?

There are several ways to automatize deduction in geometry. The one which is investigated here is to extend the basic objects: *i.e.* lines and points, of some geometric theory. The playground is the complex plane projective geometry [1,6]: only incidence properties are considered, two distinct lines always meet in one point, two distinct proper conics always meet in four points. Since Pappus theorem will be used as the main axiom, let us call it the Pappus geometry.

The main idea is to see the Pappus geometry as a functor:

- its input are two types, point and line, which fulfill axioms A_1, A_2, A_3 (given below) of the Pappus geometry; the most important axiom is Pappus property, A_3; at the first time, points and lines are the basic, naive, ones; they can be seen as symbols. It is well known that, due to the symmetry of axioms involving points and lines, points and lines can be exchanged; it is the principle of duality.

- its output is a *theory.* A theory is a set of lemmas or theorems (Desargue, Pascal, the 3-circle theorem, the 4-circle theorem, etc), their proofs, new objects (like circles and conics), and proved algorithms (drawing the conic defined by five points; computing with the ruler only the second intersection point of a line and a conic, knowing the first intersection point; etc).

It turns out that some of these new objects (*e.g.* pair of inverse points, or conics through three fixed points) generated by the theory can be considered as points and lines, actually *are* points and lines, in the sense that they comply with axioms for points and lines of the Pappus geometry.

Thus the Pappus functor can be applied a second time on these new points and lines, which are no more the naive points and lines. But the previous theory still holds, its proofs and algorithms are still valid: it will generate new theorems (or extend existing ones) and new objects. This time the generated "conics" will be cubics or quartics; in spite of their higher degree, they are still defined with five points.

P. Schreck, J. Narboux, and J. Richter-Gebert (Eds.): ADG 2010, LNAI 6877, pp. 132–151, 2011.

Again, some of the new objects can be considered as points and lines, because it is (or it should be, see below) a theorem in the Pappus theory. Thus we can apply the Pappus functor a third time. And so on.

In passing, note the similarity with compilers bootstrapping, *i.e.* compilers able to compile themself. The latter is an evidence of correctness and power of compilers.

If this approach can be formalized, say in Coq[1], it would give a way to automatically generate an infinity of non-trivial theorems. Up to now, Coq only proves already known theorems, it does not produce new ones. Also, if a dynamic geometry program can be automatically extracted from this Coq software, this dynamic geometry program would account for extended points and lines (contrarily to current dynamic geometry softwares).

However, this approach imposes constraints on the Pappus theory: its proofs must rely only on explicit axioms of the theory, and not on implicit axioms like properties of naive points and lines, which should not be shared by non naive points and lines. In principle, axiomatic geometry should satisfy this constraint, by definition of the axiomatic approach... However, some theorems in projective geometry may have no such proof for the moment: it is often easier to find algebraic proofs (with Gröbner bases, Chou's method, etc) and these methods assume properties (*e.g.* that conics are second degree algebraic curves) or coordinates which no more hold for generalized points and lines. Second, Wu remarked in his pioneering book [9] that classical proofs often neglect degeneracies. Also, maybe some axioms are missing in the Pappus theory summarized in §2, but only a formal implantation of Pappus theory, in Coq or another proof assistant, will permit to detect the gaps. To give an idea, a possible missing axiom could be: if a, b, c, d are four distinct points, not three on the same line, then the three intersection points $ab \cap cd, ac \cap bd, ad \cap bc$ are distinct and not on the same line. Or it could be some "trivial" matroid axiom which is missing.

Other predictable difficulties for an implementation in Coq are subtleties or degeneracies which are neglected in this article: it focuses on the big picture.

Plane. §2 summarizes Pappus theory. Pappus theory considers only combinatorial properties, *i.e.* incidence theorems, like Pappus, Desargue, Pascal, etc. §3 defines three times constrained conics (TTCC), and show that they can be considered as lines. However, this proof does not lie in the Pappus theory: it does not rely only on axioms A_1, A_2, A_3 of the Pappus theory. An hexamys proof (see [4] for examples of such proofs) would; but I have no such proof for the moment. §4 give some standard constraints for a conic to be a circle, a parabola, etc. §5 presents several examples of TTCC. §6 illustrates how the Pappus functor may extend theorems on non naive lines or conics. §7 sketches the generalization of points. §8 presents several variants of planes, each of which manages degeneracies (the issues of parallel lines, points at infinity, non intersecting conics, etc) in its own way. §9 concludes.

Some TTCC, and the fact they satisfy Pappus, Pascal, or Desargue' theorems, are illustrated in GeoGebra files available on internet[2].

[1] `http://coq.inria.fr/`

[2] `http://math.u-bourgogne.fr/michelucci/OCAML/GEOGEBRA/`

2 Pappus Geometry: A Summary

Pappus geometry is seen as a functor which takes two arguments, a type for lines and a type for points. We do not know what are really lines and points, we only know that they fulfil three axioms:

A_1. Two distinct points define one line.

A_2. Two distinct lines meet in one point.

A_3. If three distinct points p_i, $i = 1, 2, 3$, lie on a common line P, and three distinct points q_1, q_2, q_3 lie on a common line Q, with P and Q distinct, then the three intersection points $p_i \cap q_j$, $i \neq j$, lie on a common line.

A_3 could be called Pappus axiom.

Remark about A_2: the complex projective plane is considered; it is the set of 3D complex lines incident to a given point, say the origin: this model does not require points at infinity, so axioms do not have to consider or distinguish them. It is only for the visualization of the (real part of the) projective plane that this set of 3D lines is cut with any (affine) plane not passing through the origin; points at infinity are introduced for the 3D lines which are parallel to the cutting plane.

Pappus theory can now unfold from these three axioms.

Pappus axiom permits first to define projectivities between lines; a projectivity γ from l to l' is defined by three pairs $(p_i \in l, p'_i = \gamma(p_i) \in l')$, where $i \in 1, 2, 3$. The axis of the projectivity γ is the line through the three intersection points $p_i p'_j \cap p'_i p_j$, $i \neq j$, which are aligned after Pappus' theorem. Let x, y be two points on l and $x' = \gamma(x), y' = \gamma(y)$; then $xy' \cap x'y$ lies on the axis of the projectivity. It permits to construct the image by γ of any point x on l, assuming three pairs $(p_i, p'_i = \gamma(p_i))$.

Coxeter's book [1] provides combinatorial proofs of classical projective geometry theorems, which rely only on properties of projectivities. His book also provides algebraic proofs, using computations on cartesian or homogeneous coordinates or cross ratios.

By duality, it is possible to define a projectivity between two bundles L and L' of lines; a bundle of lines is the set of all lines passing through a common point. The projectivity is defined by three pairs of lines $(l_i \in L, l'_i \in L')$. A dual construction permits to draw with the ruler only the image of any line of L.

One of the first theorems involves the harmonic conjugate.

Harmonic Conjugate Theorem. *Let O, A, B be three aligned points. The harmonic conjugate M of O, relatively to A and B, may be constructed in many ways, using an auxilliary point S not on the line OAB, and a second auxilliary point T on SA (T, S, A are distinct). Whatever S and $T \in SA$, M is fixed, and depends only on O, A, B. If O is a point at infinity, M is the middle of A and B. This theorem is illustrated in Fig. 5 and 9.*

Projectivities can be generalized to homographies. An homography is defined by four pairs of non aligned points and their images (p_i, p'_i), with $i = 0, 1, 2, 3$. Homography of a line is a line, and the restriction of the homography to a line and its image is a projectivity. Define $l_{ij} = p_i p_j$, $l'_{ij} = p'_i p'_j$ for $i, j \in 0, 1, 2, 3$. Then the image of l_{ij} is l'_{ij}, the image of $l_{ij} \cap l_{rs}$ is $l'_{ij} \cap l'_{rs}$, etc. It is possible to draw with the ruler only the image of any point of the plane by the homography.

Another result in Pappus theory is due to Hessenberg, who proved that Desargue' theorem is a consequence of A_1, A_2, A_3:

Desargue's Theorem $(o, p_1, p_2, p_3, q_1, q_2, q_3)$**.** *Three lines l_i, $i = 1, 2, 3$, concur at o, and points o, p_i, q_i lie on l_i. Triangles $p_1p_2p_3$ and $q_1q_2q_3$ are said to be perspective (viewed from o). Then the three intersection points between homologous sides $p_ip_j \cap q_iq_j$ (with $i \neq j$) lie on a common line.*

Other theorems of Pappus theory involves conics. Of course we have first to define conics. A possible definition uses Pascal's theorem:

Here is a first definition of conic. Let p_0, p_1, p_2, p_3, p_4 be five points, no four on a common line. Then p_0, p_1, p_2, p_3, p_4 define a unique conic, which is the set of points p_5 such that the three points $p_0p_1 \cap p_3p_4$, $p_1p_2 \cap p_4p_5$, and $p_2p_3 \cap p_5p_0$ lie on a common line.

Raymond Pouzergues reformulates Pascal's theorem eliminating any reference to conics. He calls this the hexamys theorem (a shortcut for Pascal's "mystical hexagram").

Hexamys Theorem $(p_0, p_1, p_2, p_3, p_4, p_5)$**.** *Six points $p_0, p_1, p_2, p_3, p_4, p_5$ (no four colinear) are an hexamys if, by definition, opposite sides cut in three points along a common line. The hexamys theorem states that all permutations of an hexamys are hexamys as well.*

Hexamys theorem can be derived from Pappus [4].

Remark: when points $p_0p_2p_4$ lie on a common line, and points $p_1p_3p_5$ lie on another common line, then $p_0, p_1, p_2, p_3, p_4, p_5$ is an hexamys: the three intersection points of opposite sides $p_0p_1 \cap p_3p_4$, $p_1p_2 \cap p_4p_5$, and $p_2p_3 \cap p_5p_0$ lie on a common line after Pappus property. Thus pairs of distinct lines are conics.

The hexamys theorem enables Pouzergues to prove a bunch of incidence theorems: from collinearities of a given geometric configuration, the hexamys theorem deduces new collinearities. Proofs are very short [4]. Moreover, these proofs lie in the Pappus theory, *i.e.* they remain valid when naive points and lines are replaced by non naive ones : the hexamys proofs only use Pascal theorem, which is provable with Pappus theorem. For example, hexamys prove Desargue, and the harmonic conjugate theorems.

Pouzergues gives another definition of conics. Define an involution α on a line l: this involution is defined by four colinear points a, a', b, b' on l such that $\alpha(a) = a'$, $\alpha(b) = b'$. Define two distinct points u, v not on l. The set of points p such that $\alpha(up \cap l) = vp \cap l$ is a conic. Intuitively, l can be seen as the vanishing line of the plane (or the line at infinity), thus points on l are directions, and $x' = \alpha(x \in l)$ is a the direction "orthogonal" to x.

A third definition of conics can be useful. If L and L' are two bundles of lines (a bundle of lines is a set of lines all passing through a common point) in homographic bijection β: $\beta(l \in L) = l' \in L'$, then the set of intersection points $l \cap \beta(l)$ is a conic.

Hexamys permit to prove the equivalence of all these definitions of conics.

Some special conics are circles. It turns out that circles are just conics which pass through 2 special points. Classically, these 2 points are called the cyclic points, and they are often represented with homogeneous coordinates (x, y, h) equal to, for instance, $(1, \sqrt{-1}, 0)$ and $(-1, \sqrt{-1}, 0)$. Circles with center $(x_c, y_c, 1)$ and radius r have equations $x^2 + y^2 + 2x_cxh + 2y_cyh + h^2(x_c^2 + y_c^2 - r^2) = 0$, which are satisfied by cyclic points, whatever x_c, y_c, r. However, cyclic points may be replaced by any pair of distinct points, and all combinatorial theorems (which do not mention metric properties, like angles or distances) still hold. For instance this theorem.

Three Circles Theorem. *(Fig. 10, 11, 12). Let a, b, c be three points, not on a common line. a' is a point on line (bc), b' is a point on line (ac), c' is a point on line (ab). Let C_a be the circle circumscribed (CC) to a, b', c', C_b the CC to b, a', c', and C_c the CC to c, a', b'. Then C_a, C_b, C_c have a common point (other than the 2 cyclic points).*

A short proof is given in §6.1, but this proof does not lie in Pappus theory, *i.e.* this proof is not precise enough to guarantee that it follows strictly from the axioms of Pappus theory. A proof inside Pappus theory would apply to generalized lines and circles.

A theory also provides algorithms.

An algorithm to draw a conic point by point relies on Pascal theorem. Let a, b, c, d, e five points defining a conic. Let $k = ab \cap ed$. Let D a line through k. Define $i = bc \cap D$ and $j = cd \cap D$. Then $x = aj \cap ie$ lies on the conic. When D rotates around k, x draws the conic. To prove the correctness of this method, just remark that $abcdex$ is an hexamys.

Pascal's theorem also gives an algorithm to find the second intersection point between a line and a conic, passing through five points a, b, c, d, e. We want the second intersection point between az and the conic. Define $k = ab \cap ed$, $j = az \cap cd$, $D = (jk)$, $i = D \cap bc$. Then the second intersection point is $az \cap ei$.

Pascal's theorem gives an algorithm (not detailed here) to find the fourth intersection point between two conics, when the three others are known. This algorithm is useful for computing the intersection point between two non naive lines, like TTCC.

3 Three Times Constrained Conics

For convenience, 2D points are represented with homogeneous complex coordinates (x, y, h). Define

$$\phi(x, y, h) = (x^2, y^2, h^2, xy, xh, yh)$$

A conic equation is $\phi(x, y, h) \ . \ Q = 0$ where Q is a non zero vector in a $\mathbb{C}$ vector space with dimension 6 (the Hermitian scalar product is noted $\ .\ $). Each time a conic Q is constrained to pass through a point $p = (x, y, h)$, it imposes a constraint on the vector Q (the same name is used for the conic and its representing vector): $\phi(x, y, h) \ . \ Q = 0$, *i.e.* the vector Q must be orthogonal to

$\phi(p)$. Of course, the vector Q is determined, up to its norm, by five independent orthogonality conditions, thus by five points. It is consistent with the fact that conics are determined by five independent points.

But there are other constraints than passing through a specified point, which make sense, and which give the same kind of orthogonality condition on the vector Q representing a conic.

For instance, to specify that the conic Q is a circle, the vector Q must be orthogonal to $C_1 = (1, -1, 0, 0, 0, 0)$ and to the vector $C_2 = (0, 0, 0, 1, 0, 0)$; the orthogonality with C_1 imposes that the coefficients of x^2 and of y^2 in the equation of the conic Q are equal; the orthogonality with C_2 imposes that the coefficient of xy in this equation is 0. It is also possible to specify that the conic is a parabola, or a circle orthogonal to a specified circle, or a circle with its center on a specified line. The corresponding vectors are given below, §4.

Now, let C_1, C_2, C_3 be three independent such constraints. Call a conic constrained with these three constraints a three times constrained conic, a TTCC for short. These TTCC lies in a vector space with rank three : thus TTCC are 2D *lines* (or 2D points with the duality argument). 2D lines fulfil Pappus property, thus TTCC also. QED.

Unfortunately, the previous proof does not lie in the Pappus theory (it does not use only axioms A_1, A_2, A_3, it uses properties of vector spaces). A proof in the Pappus theory (for instance, an hexamys proof) would permit to apply the Pappus functor on TTCC considered as lines.

A last remark. The previous proof suggests that cubic curves constrained with 7 independent constraints, *e.g.* to pass through 7 specified points, could also be considered as lines. Since a non constrained cubic is defined by 9 (independent) points, a constrained cubic will be completely defined by two points, as naive lines; this condition is needed in order for constrained cubics to be considered as generalized lines. However:

- as for the conics, we need a definition of cubics which lie inside the Pappus theory; I think it is possible.
- two cubics must intersect in one point (the 7 constrained point do not count); this last constraint can not be satisfied: non constrained cubics cut in 9 points, after Bézout theorem; subtracting the 7 constraints, constrained cubics cut in 2 points, not 1.

More generally, which degree d algebraic curves can be considered as extended lines ?

The equation vector of an algebraic curve with degree d has $e = (d+1)(d+2)/2$ coordinates; it is a vector in a vectorial space of dimension (and rank) e. It is defined by $e-1$ constraints, *e.g.* $e-1$ points lying on the curve. Assuming the corresponding generalized line exists, it is defined by $e - 3$ fixed points (or other constraints); moreover two generalized lines must cut in just one point, ignoring the $e - 3$ fixed points; it means the two generalized lines meet in total at $e - 2$ points; but, after Bézout theorem, two degree d curves meet in d^2 points. Thus the degree d must fulfil: $d^2 = e - 2 \Leftrightarrow d^2 - 3d + 2 = 0 \Leftrightarrow d = 1$ or $d = 2$. So only algebraic curves with degree one or degree two can be considered as generalized lines.

There is here an apparent paradox, which may confuse the reader. The Pappus functor, when applied to TTCC lines, will generate new "conics", which will be cubics or quartics, and, if constrained three times, these curves can be considered as lines... The solution to this apparent paradox is that lines which feed the Pappus functor: naive lines, then TTCC, etc always lie in a vector space with *rank* three, even when the dimension is greater than three (*e.g.* six for TTCC).

4 Conditions, or Vector-Based Constraints for Conics

For short, vector-based constraints for conics are called conditions.

	x^2	y^2	h^2	xy	xh	yh
C_1	1	-1	0	0	0	0
C_2	0	0	0	1	0	0
$\phi(+1, i, 0)$	1	-1	0	i	0	0
$\phi(-1, i, 0)$	1	-1	0	$-i$	0	0
C_3	0	0	0	0	0	1
C_4	1	0	-1	0	0	1
C_5	0	1	0	0	0	0
C_6	1	0	0	0	0	1
$\phi(p) = C_p$	x_p^2	y_p^2	h_p^2	$x_p y_p$	$x_p h_p$	$y_p h_p$

Fig. 1. Possible constraints on a conic vector Q. i is $\sqrt{-1}$.

Let $Q = (a, b, c, d, e, f)$ be the vector representing a conic. The equation of the conic is $ax^2 + by^2 + ch^2 + dxy + exh + fyh = 0$. This section gives possible constraints on the conic, they are summarized in table 1.

The conic passes through a point $p = (x_p, y_p, h_p)$ if Q is orthogonal to the vector $C_p = \phi(p)$.

The conic is a circle if Q is orthogonal to C_1 and C_2. Orthogonality to C_1 implies that $a = b$, orthogonality to C_2 means coefficient of monomial xy is zero. Equivalent conditions are that Q passes through cyclic points $(\pm 1, i, 0)$ (with $i^2 = -1$), thus Q is orthogonal to both $\phi(\pm 1, i, 0)$.

The circle has its center on the line $y = 0$ if Q is orthogonal to C_3.

The circle is orthogonal to the unit circle with equation $x^2 + y^2 - 1 = 0$ if Q is orthogonal to C_4 (proof: see Fig. 2).

The circle cuts the unit circle (*i.e.* $x^2 + y^2 - 1 = 0$) in two points symmetric relatively to the origin $(0, 0)$ if Q is orthogonal to C_6. These circles have equations $x^2 + y^2 - 2ux - 2vy - 1 = 0$, the center is (u, v) and the radius is R such that $R^2 = 1 + u^2 + v^2$.

The conic is a parabola with axis Oy if Q is orthogonal to C_5 and C_2, *i.e.* the coefficients for y^2 and xy are 0.

Some constraints do not give orthogonality conditions, for instance the tangence of a circle Q to a prescribed line, say $y = 0$.

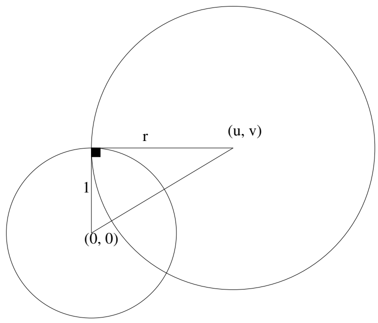

Fig. 2. The circle with center (u, v) and radius r is orthogonal to the unit circle. Thus $u^2+v^2 = 1+r^2$, after Pythagora. Its equation (in affine coordinates) is $x^2+y^2-2ux-2vy+(u^2+v^2-r^2) = 0$, *i.e.* $x^2+y^2-2ux-2vy+1 = 0$. Thus the coefficient for the constant must equal the coefficient for x^2, and for y^2 in the homogeneous equation.

5 Examples of Non Naive Lines

§5.1 shows that circles through a given fixed point can be considered as lines. §5.2 shows that circles orthogonal to a given fixed circle and passing through a given fixed point can be considered as lines. §5.3 shows that circles (or half circles) with their centers lying on a given fixed line can be considered as lines. §5.4 shows that circles which cut the unit circle in two points symmetric relatively to the origin can be considered as lines. §5.5 shows that parabolas with axis parallel to a given fixed direction and passing through a given fixed point can be considered as lines. §5.6 shows that conics passing through three given fixed points can be considered as lines.

5.1 Circles through One Fixed Point

Let Ω be a fixed, arbitrary, point. Then circles (in the classical sense) through Ω can be considered as lines. For convenience, such circles are called clines in this section. Two distinct clines cut in one point (ignoring Ω and the two cyclic points); it can happen that Ω is a double intersection point; in this case, one may say that the two clines are parallel, and that they meet at a point at infinity, which is Ω. Two distinct points (and distinct of Ω) define an unique cline. Clines satisfy the Pappus property, as illustrated in Fig. 3.

Clines satisfy Pappus property: *i.e.* if p_0, p_1, p_2 lie on a common cline, and q_0, q_1, q_2 lie on another common cline, then the three intersection points r_{ij} between the cline p_iq_j and p_jq_i, $i \neq j$, lie on a common cline.

It has already been proved, but this new proof may be instructive. An inversion relatively to any circle (say with radius 1) with center Ω maps points p_i to point p'_i, points q_j to points q'_j, and points r_{ij} to points r'_{ij}, and it maps clines to naive lines not passing through Ω. Thus the points p'_i, q'_j, and r'_{ij} satisfy the Pappus property, *i.e.* the intersection points r'_{ij} lie on the same line, call it R'. The preimage of R' by the inversion is a cline R; in the peculiar case where R'

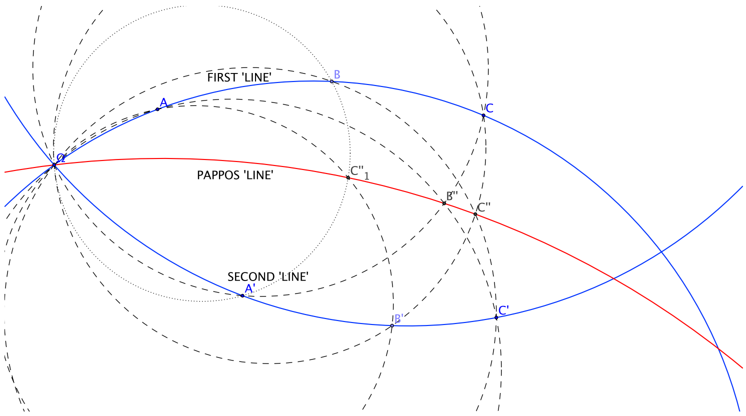

Fig. 3. Clines fulfil Pappus property. They can be considered as lines.

pass through Ω, its preimage is $R = R'$, so it is a (degenerate) cline. In all cases, the preimage R of R' is a cline, thus the r_{ij} lie on a common cline, R. QED.

Thus all theorems of Pappus theory still hold when the word "line" is replaced by the word "cline". For instance the hexamys theorem holds. Define a C-hexamys as a set of six points, no four on the same cline, such that opposite clines meet in three points lying on a common cline. Then any permutation of the six points is also a C-hexamys.

Fig. 4 illustrates Pascal's theorem with clines. For simplicity, the six points lie on a common circle (which does not pass through Ω). The three pairs of opposite clines indeed lie on a common cline, *i.e.* they are cocyclic with Ω.

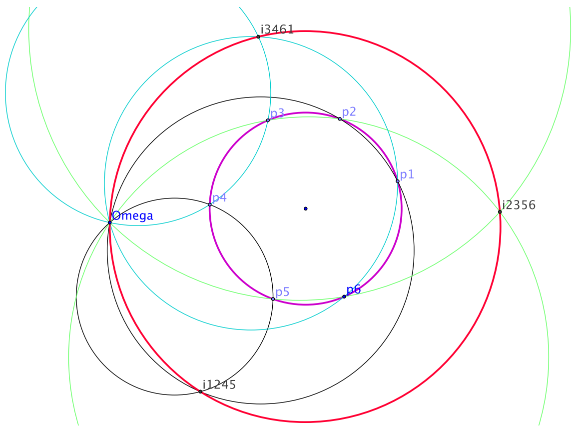

Fig. 4. Pascal theorem. Points p_i lie on the magenta circle. The lines p_ip_j are replaced with clines (circles through Omega). The intersection points lie on a common cline (red circle).

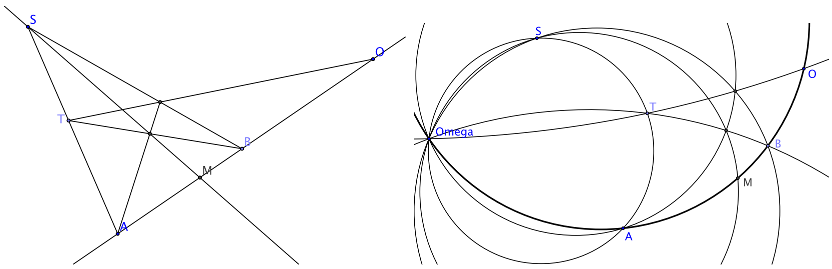

Fig. 5. The harmonic conjugate theorem. Left: for given points O, A, B on a common line, for any point S, for any point T on the line SA, the point M is invariant (hint: M is the harmonic conjugate of O relatively to A, B; if O is a point at infinity, M is the middle of AB. Right: all lines are replaced with clines, M is still invariant.

Fig. 5, Right, illustrates the harmonic conjugate theorem with clines.

What are conics in the Pappus of clines ? They are images of a naive conic by an inversion, thus they are quartic curves, or cubic curves in degenerate cases (the inversion center lies on the conic).

Remark. In the projective complex plane, the inversion is not defined on Ω. It can be defined for other planes (§8). These details are predictable sources of complications for a Coq implementation.

5.2 Orthogonal Circles

Circles orthogonal to a given fixed circle can be considered as lines. A difficulty is due to the fact that such circles cut in two points. These two points are inverse of each other and always come in pairs. Thus it is sufficient to consider these pairs as generalized points. Another solution is to consider only one side (either the inside, or the outside) of the given fixed circle.

5.3 Poincaré Half Circles Are Lines

Circles the centers of which lie on a given line, for example $y = 0$, can be considered as lines. Fig. 6 illustrates the Pappus property for these generalized lines. Points come in pairs, with a symmetry relatively to the line $y = 0$. To define related generalized points, either only points and half circles above the line $y = 0$ are considered, or pairs of symmetric points are considered.

In passing, the Poincaré model for the hyperbolic plane uses these half circles, it is called the Poincaré half plane [8] (curiously, this book does not mention the Pappusian feature of the Poincaré plane).

5.4 Other Circles

Circles which cuts the unit circle (having equation: $x^2 + y^2 - 1 = 0$) in two points symmetric relatively to the origin $(0, 0)$ can also be considered as generalized

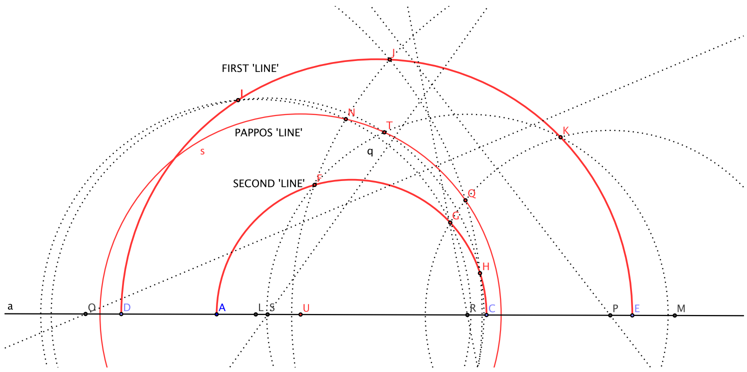

Fig. 6. Circles with centers on a common line (*e.g.* $y = 0$) fulfil Pappus property. Points come in pairs.

lines. Like all circles, their vector Q is orthogonal to C_1 and C_2; moreover their vector Q is orthogonal to C_6. They have equations $x^2+y^2-2ux-2vy-1=0$, their center is (u, v), their radius is $\sqrt{1+u^2+v^2}$. Two distinct circles in this family always meet in two antipodal points of the unit circle. Fig. 7 shows a bundle of such circles. It illustrates the fact that all these circles belong to a bundle generated by the unit circle $x^2+y^2-1=0$ and a line with equation $ux+vy=0$. Thus all circles of this bundle pass through points $(v/\sqrt{u^2+v^2}, -u/\sqrt{u^2+v^2})$ and $(-v/\sqrt{u^2+v^2}, u/\sqrt{u^2+v^2})$.

Points for these generalized lines are pair of naive points, which are symmetric w.r.t. the origin.

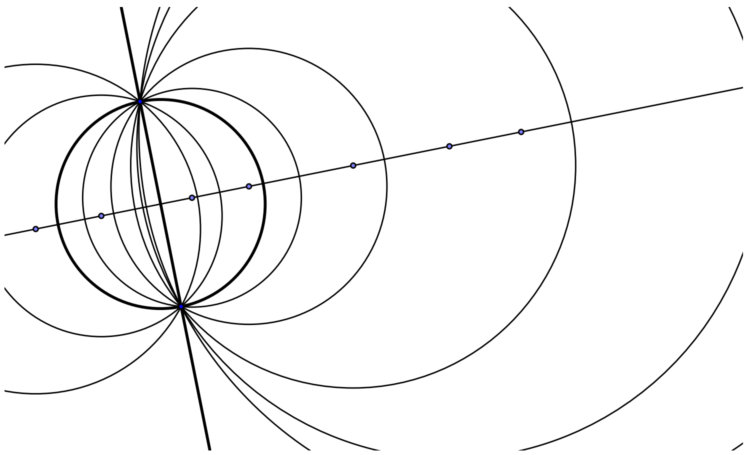

Fig. 7. A bundle of circles. The thick circle and the thick line generate the bundle. The full class of these circles is obtained when rotating the line.

Another way to generate circles in this class is to compose two projections; it also gives another proof of the fact that these circles are generalized lines; first project naive lines in the plane to great circle on a 3D sphere, with the center of the sphere as the center of projection. This projection maps each point of the plane to two antipodal points on the sphere, which are equivalent. Then apply a stereographic projection from the sphere to the (say, equatorial) plane, *i.e.* the center of the projection is a pole of the sphere. The proof relies on easy but tedious computations which are omitted for conciseness. Both projections preserve incidences, thus the Pappus property holds for great circles on the sphere, and for the final circles.

These circles are lines in the Beltrami model of the hyperbolic plane [8,2].

5.5 Some Parabolas Are Lines

Parabolas with a prescribed axis direction (say Oy) and passing through a given fixed point can be considered as lines. They are completely defined with two other points, like naive lines. These parabolas cut in at most one point (ignoring the fixed common point, and the double point at infinity: $(0, 1, 0)$). As usual, two parabolas non intersecting in the affine real plane do meet in the projective complex plane.

5.6 Conics through Three Fixed Points Are Lines

The GeoGebra figure 8 illustrates that conics through three given distinct points (non colinear) can be considered as lines: they fulfil A_3, the Pappus axiom. They also fulfil A_1 and A_2. Fig. 9 illustrates the harmonic conjugate theorem.

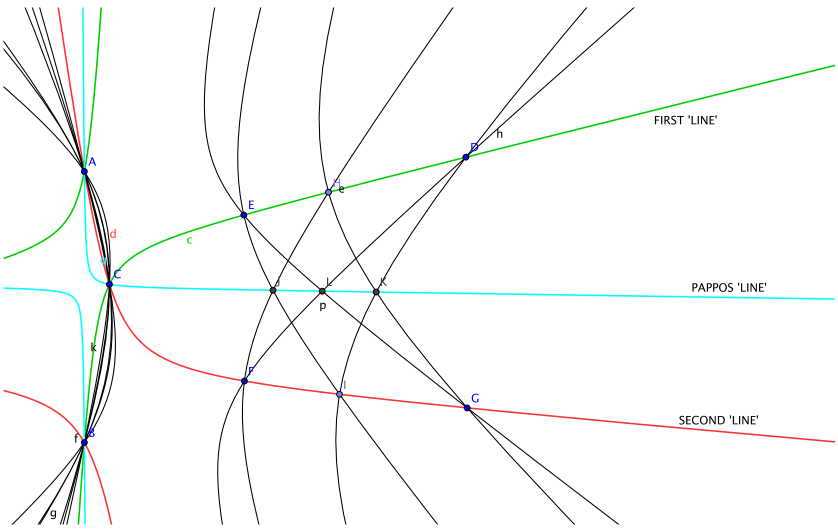

Fig. 8. Conics passing through three given distinct points ($A, , B, C$ on the figure) fulfil Pappus axioms. Thus they can be considered as lines.

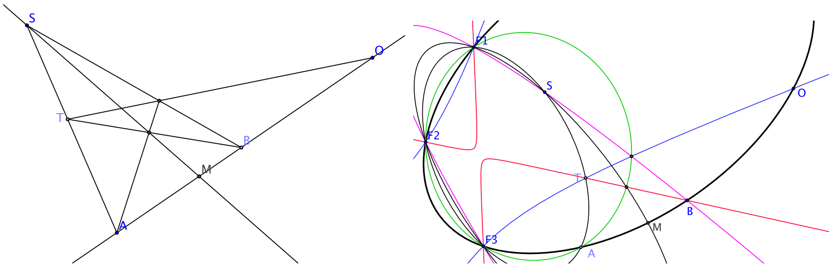

Fig. 9. The harmonic conjugate theorem. Left: the harmonic conjugate theorem for naive lines. Right: the harmonic conjugate theorem for conics passing through three fixed points F_1, F_2, F_3.

6 Playing with Some Theorems

This section illustrates how a Pappus functor may extend theorems, on three examples.

6.1 Proof of the Three Circles Theorem

The three circles theorem is used as an example of a theorem, for which I know no proof lying in the Pappus theory for the moment.

Three Circles Theorem. *Let a, b, c be three points, not on a common line. a' is a point on line (bc), b' is a point on line (ac), c' is a point on line (ab). Let C_a be the circle circumscribed (CC) to a, b', c'. C_b is the CC to b, a', c', and C_c is the CC to c, a', b'. Then C_a, C_b, C_c have a common point (other than the two cyclic points).*

A short proof is given here, but it does not lie inside Pappus theory. A proof inside Pappus theory would permit to extend this theorem to generalized lines.

The proof considers lines. The lines of the triangle are indexed 1, 2, 3, see Fig. 10 for the definition of lines 5,6,7. By hypothesis, the points $1 \cap 2, 2 \cap 4, 4 \cap 5, 5 \cap 1$ are cocylic, as well as the points $5 \cap 6, 6 \cap 3, 3 \cap 1, 1 \cap 5$. We need to prove that the points $2 \cap 3, 3 \cap 6, 6 \cap 4$, $4 \cap 2$ are cocyclic too. Note 1, 2,. . . 6 the orthogonal symmetry relatively to line 1, 2, . . . 6.

We first need the lemma: the transform 5124 is a translation. I use the convention that in the transform 5124, the symmetry 5 is performed first, but anyway it does not matter: the reader can uses the opposite convention when reading the proof. In the transform $5124 = (51)(24)$, the transforms 51 and 24 are rotations; 51 is a rotation around $5 \cap 1$, with angle twice the angle between lines 5 and 1. Similarly, 24 is a rotation around $2 \cap 4$, with angle twice the angle between lines 2 and 4. But opposite angles in a cocyclic quadrilateral are either opposite, or their sum equals π. In both cases, the effect of rotations 51 and 24 on vectors annihilate each other, so 5124 is just a translation. QED. The converse also holds.

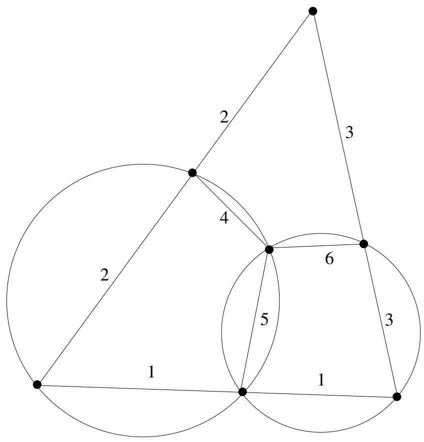

Fig. 10. The three circles theorem. The three circles have a common point.

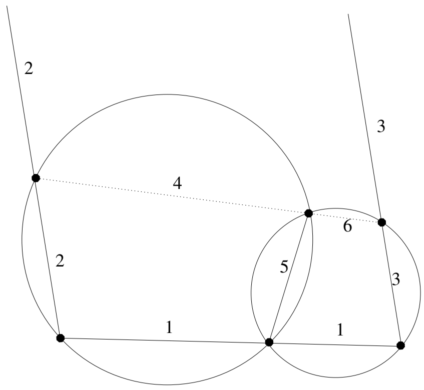

Fig. 11. Here we do not know that lines 4 and 6 are equal, we have to prove it. As in the generic case, 6315 and 5124 are translations. Thus their composition $(6315)(5124) = 6324$ is a translation too. Circular permutations of a translation are translations too [3], thus 3246 is a translation too. Moreover 32 and its inverse 23 are translations because lines 2 and 3 are parallel in this special case. Thus $(23)(3246) = 46$ is a translation. Thus lines 4 and 6 are parallel. But they have a common point ($6 \cap 5$ and $4 \cap 5$), thus they are equal. QED.

Similarly, 6315 is a translation.

Thus the composition $(6315)(5124) = 63(1(55)1)24 = 6324$ is a translation, thus the four points $6 \cap 3, 3 \cap 2$, $2 \cap 4$, $4 \cap 6$ are cocyclic. It is worth to mention that this proof works also when the triangle 1, 2, 3 is degenerate, *e.g.* when lines 2 and 3 are parallel, as in Fig. 11.

Actually the three circles theorem still holds when circle are replaced with conics passing through two distinct arbitrary points. See Fig. 12.

Another correct generalization of the three circles theorem is illustrated Fig.13. It replaces Euclidean lines with conics passing through three distinct fixed (non

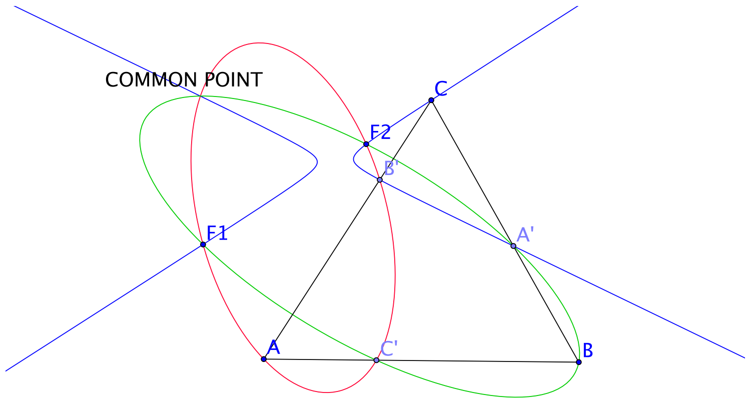

Fig. 12. A generalization of the three circles theorem. Circles are replaced with conics passing through 2 distinct arbitrary points F_1, F_2. These three conics have a common point (other than the two arbitrary points).

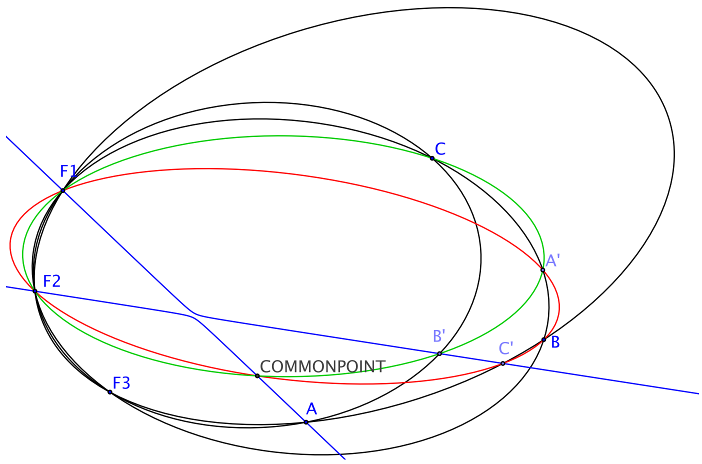

Fig. 13. An extension of the three circles theorem. Lines (AB, AC, BC) are replaced with conics passing through three fixed points F_1, F_2, F_3, and circles are replaced with conics through two of the fixed three points, for instance F_1 and F_2. The three generalized circles have a common point, different of F_1 and F_2.

aligned) points F_1, F_2 and F_3, and replaces circles with conics through F_1 and F_2. Then the three "circles" have a common pont.

A Pappus functor should be able to automatically produce such non trivial generalizations of the three circles theorem and the corresponding proofs.

6.2 The Four Circles Theorem

Four Circles Theorem. *It is also called Miquel's four circles theorem. It states that the four circles circumbscribed to three points of a complete quadrilateral have a common point, see Fig.14.*

I know no proof in Pappus theory up to now (a combinatorial search for hexamys by computer should find one). Anyway, the theorem can be proved easily from Chasles theorem: each circle union the "opposite" line defines a cubic curve; the four cubic curves meet in 8 common points: the two cyclic points and the six points of the complete quadrilateral. Thus after Chasles theorem, these cubics meet in another nineth point.

Another short and nice proof relies on orthogonal symmetries relatively to lines of the complete quadrilateral, see Fig.14 for the names of the lines. By hypothesis, $ACUH$ is cocyclic, thus the transform $ACUH$ is a translation. Idem for $HVDA$. Thus the composition $(ACUH)(HVDA) = ACUVDA$ is a translation as well. Thus $A(ACUVDA)A = CUVD$ is a translation too. Thus $CUVD$ is cocyclic. QED. Unfortunately this last proof can not be generalized to generalized lines.

A Pappus functor should be able to produce this non trivial generalization (Fig. 15) of the four circles theorem. Let F_1, F_2, F_3 be three distinct non aligned points. The six points (which were the vertices of the complete quadrilateral in the initial four circles theorem) are called $Q_i, i = 1, \dots 6$, and there are four conics. The conic K_{134} passes through points $F_1, F_2, F_3, Q_1, Q_3, Q_4$; the conic K_{156} passes through points $F_1, F_2, F_3, Q_1, Q_5, Q_6$; the conic K_{235} passes through points $F_1, F_2, F_3, Q_2, Q_3, Q_5$; the conic K_{246} passes through points $F_1, F_2, F_3, Q_2, Q_4, Q_6$. Replace circles in the initial four circles theorems with conics passing through points F_1 and F_2. Then the four conics: $K_{134}, K_{156}, K_{235}, K_{246}$ all pass through another common point, Z in Fig. 15.

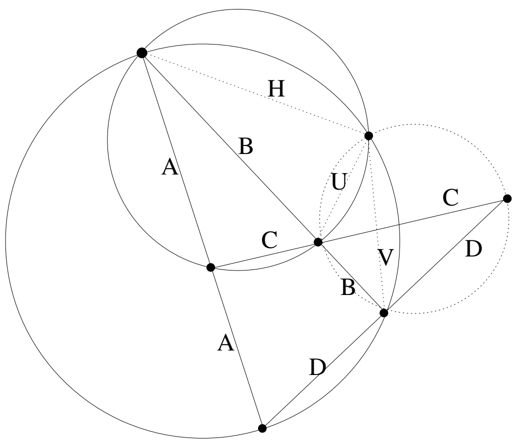

Fig. 14. Miquel's four circles theorem: the four circles have a common point (distinct of the two cyclic points)

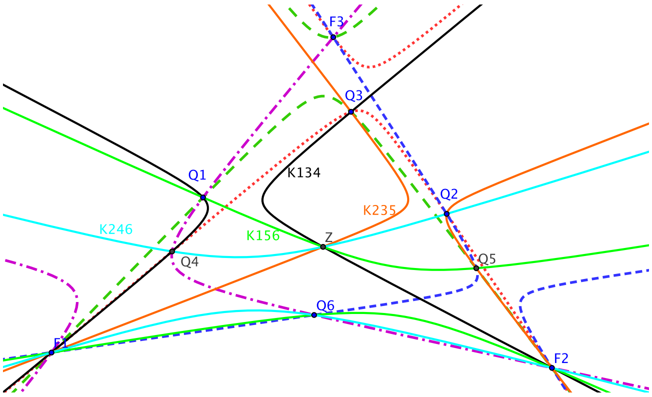

Fig. 15. An extension of Miquel's four circles theorem: lines are replaced with conics through three fixed points F_1, F_2, F_3, circles are replaced with conics through two of the fixed points, namely F_1 and F_2. Then the four generalized circles have a common point, Z in the figure.

6.3 A Butterfly Theorem

We conclude with this last theorem, Fig. 16. Let C be a fixed circle, and E a point not on C. The symmetric to a point $M \in C$ is by definition $M' = (EM) \cap C$. It is clearly an involution. The symmetry is extended to all points in the plane with a Butterfly theorem which states that for all chords (A_1, A_2) through M (where $A_1 \in C, A_2 \in C$), the symmetric chords (A'_1, A'_2) passes through a common point, which is M'. Any conic can be used in place of the circle C (for instance two lines, which gives a variant of Pappus theorem), and the theorem still holds. For conciseness, no proof is provided. A Pappus functor should be able to generalize (Fig. 16) this theorem and its proof (if it lies in Pappus theory). A first generalization replaces linear chords with clines, *i.e.* circles through a fixed point. Since this generalization reduces to applying some inversion, it may be considered trivial. A second generalization is less obvious; it replaces lines with conics through three fixed points F_1, F_2, F_3, and the circle C is replaced with a circle (or any conic) passing through F_1 and F_2.

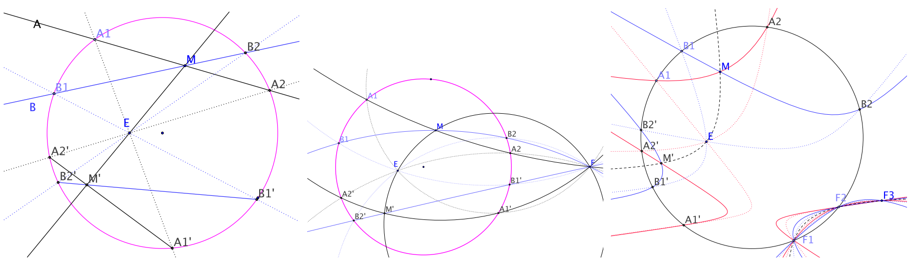

Fig. 16. From left to right: a butterfly theorem, a first generalization, and a second one

7 What Is a Point ?

This article mainly generalized lines. Another way to extend Pappus theory is to generalize points. It is well known that, due to duality, lines and points can be exchanged. It is conics which I will consider as points, in some sense.

A conic is represented with a non zero vector in a six dimensions vector space. So it can be seen as a point in a projective space in five dimensions. We can say that three conics Q_1, Q_2, Q_3 are aligned iff there are non all zeros numbers a_1, a_2, a_3 such that $a_1Q_1 + a_2Q_2 + a_3Q_3 = 0$. Two distinct conics Q_1 and Q_2 generates a "line of conics", *i.e.* the set of conics equal to $a_1Q_1 + a_2Q_2$ for some numbers a_1, a_2. To avoid ambiguity, call it a 2-bundle of conics.

Similarly, four conics are coplanar iff there are non all zeros numbers a_1, a_2, a_3, a_4 such that $a_1Q_1 + a_2Q_2 + a_3Q_3 + a_4Q_4 = 0$. Three non aligned conics generate a plane of conics, called a 3-bundle of conics.

A 3-bundle of conics is a Pappus plane, its points are conics. Thus all theorems of the Pappus plane apply: Pappus, Desargue, Pascal, three-circle theorems, etc. We can apply the Pappus functor.

8 Variants: A Zoo of Planes

For simplicity, we considered only the strongest axioms, so two distinct lines always meet in one point, and two distinct proper conics always meet in four points. It is the complex projective plane.

Weaker axioms, and other planes, are possible. For instance, we can accept that two distinct lines meet in at most one point, and that two proper conics meet in at most four points. The essential constraint is that no configuration contradicts Pappus axiom, which can be rephrased as follows: if three distinct points p_i are aligned, and if three distinct points q_j are aligned on another line, and if the three intersection points $r_{ij} = p_i \cap q_j, i < j$ exist, then they are aligned.

This freedom of choice for axioms is related to the fact that the plane can be coordinalized in several ways [6]. A point can be represented with two real cartesian coordinates (x, y): it is the affine real plane, $\mathbb{R}^2$; it contains parallel lines which do not meet. A point can be represented with homogenous real coordinates $(x, y, h) \in \mathbb{R}^3 \setminus (0, 0, 0)$, two colinear vectors representing the same point; this representation can be made canonic, using only values zero and one for the homogeneous coordinate h; points $(x, y, 0)$ are points at infinity; this is the real projective plane $P^2(\mathbb{R})$; all pair of distinct lines meet in one point; but two distinct proper conics can meet in less than four points because $\mathbb{R}$ is not algebraically closed. Geometrically, $P^2(\mathbb{R})$ is the set of 3D lines incident to a given point, say the origin; to visualize the plane, the set of lines is cut with an arbitrary plane not passing through the origin. To get more regularity, a solution is to use complex coordinates, either cartesian coordinates $(x, y) \in \mathbb{C}^2$, or homogeneous coordinates $(x, y, h) \in \mathbb{C}^3 \setminus (0, 0, 0)$, *i.e.* $P^2(\mathbb{C})$. Another classical representation represents each point $(x \in \mathbb{R}, y \in \mathbb{R})$ of the plane with a complex number $c = x + iy \in \mathbb{C}$: this is the complex line $\mathbb{C}$; if $\mathbb{C}$ is augmented with $1/0$ for convenience, the complex projective line $P^1(\mathbb{C})$ is obtained: this plane has only one point at infinity; through stereographic projection, this plane is mapped to the sphere $\mathcal{S}^2$ ($\mathcal{S}^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$) and its point at infinity $1/0$ is mapped to the North pole $(0, 0, 1)$ of the sphere. All these planes are locally equivalent for their "visible" part, but they are no more when points at infinity or imaginary points are involved; also they have not the same topology.

Example. Consider the circle with center $(-3, 0)$ and radius 1, and the circle with center $(3, 0)$ and radius 1. In the affine real plane $\mathbb{R}^2$, in the projective real plane $P^2(\mathbb{R})$, and in the complex projective line $P^1(\mathbb{C})$, they do not intersect. In the complex affine plane $\mathbb{C}^2$ (points are represented with $(x \in \mathbb{C}, y \in \mathbb{C})$), they intersect in two points: $(0, \pm i\sqrt{3})$. In the complex projective plane $P^2(\mathbb{C})$, they intersect in four points, the two previous ones, and the cyclic points $(\pm i, 1, 0)$.

Remark. In the complex projective line $P^1(\mathbb{C})$, inversions, for example: $T(z) = 1/\overline{z}$ and $T'(z) = 1/z$, can be extended to their pole 0: the pole and the point at infinity are inverse of each other. In the complex projective plane $P^2(\mathbb{C})$, the inversion: $T(x, y) = (x/(x^2 + y^2), y/(x^2 + y^2))$ (using cartesian coordinates for short) can not be consistently extended to the point $(0, 0)$: it is because $P^2(\mathbb{C})$ has a line at infinity, and not one point at infinity like $P^1(\mathbb{C})$.

Remark. It is convenient to map $P^1(\mathbb{C})$ to the sphere $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 - 1 = 0\}$ with the stereographic projection s. For convenience, place the plane $P^1(\mathbb{C})$: $c = x + iy$ horizontally at altitude $z = 0$; then the stereographic projection maps $c = (x, y, 0)$ to $s(c) = (2x/(x^2 + y^2 + 1), 2y/(x^2 + y^2 + 1), (x^2 + y^2 - 1)/(x^2 + y^2 + 1))$. $s(c)$ is the intersection point of the sphere and the line (Nc), where $N = (0, 0, 1)$ is the North pole of the sphere. Naive lines in $P^1(\mathbb{C})$ are mapped to circles on the sphere, all passing through N. The point at infinity of $P^1(\mathbb{C})$ is mapped to N. Some properties of $P^1(\mathbb{C})$ are more easily seen on the sphere, *e.g.* in $P^1(\mathbb{C})$, the point at infinity $1/0$ belongs to all (naive) lines; thus non parallel lines (in the usual, naive sense) in $P^1(\mathbb{C})$ cut in two points.

The "Pappus tower" can likely be built with these planes. However, each of them manages degeneracies (parallel lines, non intersecting conics, points at infinity) in its own way, which may complicate implementations.

9 Conclusion

Two remarks before concluding:

- From another viewpoint, the content of this article is sometimes trivial. We just apply many homographies and inversions to the whole naive Pappus plane, so naive lines and naive conics are mapped to curves with arbitrary high degree. What is essential is that all these transforms (homographies and inversions) preserve incidences. More general non linear diffeomorphisms could be used as well.
- Jürgens Richter-Gebert et al [5] show that tropical lines do not always fulfil Pappus property.

In conclusion, this article considers the Pappus theory as a functor: its inputs are points and lines which must fulfil axioms of Pappus geometry. The output is a set of proved theorems and methods, and new geometric objects, some of which fulfil axioms of Pappus geometry. Theorems are incidence theorems, and have a combinatorial flavor.

For this approach to work in practice, *e.g.* to be programmed in Coq, all proofs must lie inside the Pappus theory, *i.e.* all proofs must use only axioms of the Pappus theory. A computer combinatorial search inspired by the area method or the full-angle method [7], but through the set of Hexamys (or their duals, Brianchons) as in [4], and relying on some numerical example (a figure, or a witness) like the area method to help prune the search space, may help find such proofs in an automatic way.

This article was written with in mind a geometric formalization, *i.e.* theorems and algorithms are proved applying the Pappus axiom, or the hexamys theorems, or relying on properties of projectivities or homographies, like in Coxeter's book [1]. However a more algebraic approach can also be considered; for instance, lines can be seen algebraically as vectors in some rank three vector space.

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